Rules for integrands involving Bessel functions

1. $\int u\,\text{BesselJ}[n,\ a+b\,x]\,dx$

   1. $\int \text{BesselJ}[n,\ a+b\,x]\,dx$

      1. $\int \text{BesselJ}[n,\ a+b\,x]\,dx$ when $\frac{n+1}{2}\in\mathbb{Z}^+$

         1: $\int \text{BesselJ}[1,\ a+b\,x]\,dx$

Rule:

$$\int \text{BesselJ}[1,\ a+b\,x]\,dx \longrightarrow -\frac{\text{BesselJ}[0,\ a+b\,x]}{b}$$

Program code:

```
Int[BesselJ[1,a_.+b_.*x_],x_Symbol] :=
  -BesselJ[0,a+b*x]/b /;
FreeQ[{a,b},x]
```

         2: $\int \text{BesselJ}[n,\ a+b\,x]\,dx$ when $\frac{n-1}{2}\in\mathbb{Z}^+$

Basis: $\text{BesselJ}[n,\ a+b\,x] == -\frac{2\,\partial_x\text{BesselJ}[n-1,a+b\,x]}{b} + \text{BesselJ}[n-2,\ a+b\,x]$

Rule: If $\frac{n-1}{2}\in\mathbb{Z}^+$, then

$$\int \text{BesselJ}[n,\ a+b\,x]\,dx \longrightarrow -\frac{2\,\text{BesselJ}[n-1,\ a+b\,x]}{b} + \int \text{BesselJ}[n-2,\ a+b\,x]\,dx$$

Program code:

```
Int[BesselJ[n_,a_.+b_.*x_],x_Symbol] :=
  -2*BesselJ[n-1,a+b*x]/b + Int[BesselJ[n-2,a+b*x],x] /;
FreeQ[{a,b},x] && IGtQ[(n-1)/2,0]
```

x: $\int \text{BesselJ}[n, a + b x] \, dx$ when $n \in \mathbb{Z}^-$

- Derivation: Algebraic simplification

- Basis: If $n \in \mathbb{Z}$, then $\text{BesselJ}[n, z] == (-1)^n \text{BesselJ}[-n, z]$

Note: This rule not necessary since *Mathematica* automatically simplifies $\text{BesselJ}[n, a + b x]$ to $(-1)^n \text{BesselJ}[-n, z]$ if $n \in \mathbb{Z}^-$.

Rule: If $n \in \mathbb{Z}^-$, then

$$\int \text{BesselJ}[n, a + b x] \, dx \longrightarrow (-1)^n \int \text{BesselJ}[-n, a + b x] \, dx$$

Program code:

```
(* Int[BesselJ[n_,a_.+b_.*x_],x_Symbol] :=
  (-1)^n*Int[BesselJ[-n,a+b*x],x] /;
FreeQ[{a,b},x] && ILtQ[n,0] *)
```

2: $\int \text{BesselJ}[n, a + b\,x]\,dx$

Rule:

$$\int \text{BesselJ}[n, a + b\,x]\,dx \longrightarrow \frac{(a + b\,x)^{n+1}\,\text{HypergeometricPFQ}\left[\left\{\frac{n+1}{2}\right\}, \left\{\frac{n+3}{2}, n + 1\right\}, -\frac{1}{4}\,(a + b\,x)^2\right]}{2^n\,b\,\text{Gamma}[n + 2]}$$

Program code:

```
Int[BesselJ[n_,a_.+b_.*x_],x_Symbol] :=
  (a+b*x)^(n+1)*HypergeometricPFQ[{(n+1)/2},{(n+3)/2,n+1},-1/4*(a+b*x)^2]/(2^n*b*Gamma[n+2]) /;
FreeQ[{a,b,n},x]
```

2. $\int (d\,x)^m\,\text{BesselJ}[n, b\,x]\,dx$

3. $\int (c + d\,x)^m\,\text{BesselJ}[n, a + b\,x]\,dx$

2. $\int u\,\text{BesselK}[n, a + b\,x]\,dx$

3. $\int u\,\text{BesselY}[n, a + b\,x]\,dx$