Rules for integrands of the form $P_q[x]\,(a+b x^2)^p$

1: $\int P_q[x]\,(a+b x^2)^p\,dx$ when $p+2\in\mathbb{Z}^+$

- **Derivation: Algebraic expansion**
- **Rule 1.1.2.x.1: If $p+2\in\mathbb{Z}^+$, then**

$$\int P_q[x]\,(a+b x^2)^p\,dx \longrightarrow \int \text{ExpandIntegrand}\left[P_q[x]\,(a+b x^2)^p,\,x\right]dx$$

- **Program code:**

```
Int[Pq_*(a_+b_.*x_^2)^p_.,x_Symbol] :=
  Int[ExpandIntegrand[Pq*(a+b*x^2)^p,x],x] /;
FreeQ[{a,b},x] && PolyQ[Pq,x] && IGtQ[p,-2]
```

2: $\int P_q[x]\,(a+b x^2)^p\,dx$ when $P_q[x,0]=0$

- **Derivation: Algebraic simplification**
- **Rule 1.1.2.x.2: If $P_q[x,0]=0$, then**

$$\int P_q[x]\,(a+b x^2)^p\,dx \longrightarrow \int x\,\text{PolynomialQuotient}\left[P_q[x],\,x,\,x\right]\,(a+b x^2)^p\,dx$$

- **Program code:**

```
Int[Pq_*(a_+b_.*x_^2)^p_,x_Symbol] :=
  Int[x*PolynomialQuotient[Pq,x,x]*(a+b*x^2)^p,x] /;
FreeQ[{a,b,p},x] && PolyQ[Pq,x] && EqQ[Coeff[Pq,x,0],0] && Not[MatchQ[Pq,x^m_.*u_. /; IntegerQ[m]]]
```

3: $\int P_q[x]\,(a+b\,x^2)^p\,dx$ when $\text{PolynomialRemainder}[P_q[x],\,a+b\,x^2,\,x]==0$

- **Derivation: Algebraic expansion**
- **Rule: If** $\text{PolynomialRemainder}[P_q[x],\,a+b\,x^2,\,x]==0$**, then**

$$\int P_q[x]\,(a+b\,x^2)^p\,dx\;\longrightarrow\;\int \text{PolynomialQuotient}[P_q[x],\,a+b\,x^2,\,x]\,(a+b\,x^2)^{p+1}\,dx$$

- **Program code:**

```
Int[Px_*(a_+b_.*x_^2)^p_.,x_Symbol] :=
  Int[PolynomialQuotient[Px,a+b*x^2,x]*(a+b*x^2)^(p+1),x] /;
FreeQ[{a,b,p},x] && PolyQ[Px,x] && EqQ[PolynomialRemainder[Px,a+b*x^2,x],0]
```

4. $\int P_q[x]\,(a+b\,x^2)^p\,dx$ when $p<-1$

1: $\int P_q[x^2]\,(a+b\,x^2)^p\,dx$ when $p+\frac{1}{2}\in\mathbb{Z}^-\bigwedge 2\,q+2\,p+1<0$

Derivation: Algebraic expansion and binomial recurrence 3b

- **Basis:** $\int (a+b\,x^2)^p\,dx = \frac{x\,(a+b\,x^2)^{p+1}}{a}-\frac{b\,(2\,p+3)}{a}\int x^2\,(a+b\,x^2)^p\,dx$

Note: Interestingly this rule eleminates the constant term of $P_q[x^2]$ rather than the highest degree term.

- **Rule 1.1.2.x.4.1: If** $p+\frac{1}{2}\in\mathbb{Z}^-\bigwedge 2\,q+2\,p+1<0$, **let** $A\to P_q[x^2,\,0]$ **and** $Q_{q-1}[x^2]\to\text{PolynomialQuotient}[P_q[x^2]-A,\,x^2,\,x]$, **then**

$$\int P_q[x^2]\,(a+b\,x^2)^p\,dx\longrightarrow$$

$$A\int (a+b\,x^2)^p\,dx+\int x^2\,Q_{q-1}[x^2]\,(a+b\,x^2)^p\,dx\longrightarrow$$

$$\frac{A\,x\,(a+b\,x^2)^{p+1}}{a}+\frac{1}{a}\int x^2\,(a+b\,x^2)^p\,(a\,Q_{q-1}[x^2]-A\,b\,(2\,p+3))\,dx$$

Program code:

```
Int[Pq_*(a_+b_.*x_^2)^p_,x_Symbol] :=
  With[{A=Coeff[Pq,x,0],Q=PolynomialQuotient[Pq-Coeff[Pq,x,0],x^2,x]},
  A*x*(a+b*x^2)^(p+1)/a + 1/a*Int[x^2*(a+b*x^2)^p*(a*Q-A*b*(2*p+3)),x]] /;
FreeQ[{a,b},x] && PolyQ[Pq,x^2] && ILtQ[p+1/2,0] && LtQ[Expon[Pq,x]+2*p+1,0]
```

2: $\int P_q[x]\,(a+b\,x^2)^p\,dx$ when $p<-1$

Derivation: Algebraic expansion and quadratic recurrence 2a

- **Rule 1.1.2.x.4.2: If** $p<-1$,
  let $Q_{q-2}[x]\to\text{PolynomialQuotient}[P_q[x],\,a+b\,x^2,\,x]$ **and** $f+g\,x\to\text{PolynomialRemainder}[P_q[x],\,a+b\,x^2,\,x]$, **then**

$$\int P_q[x]\,(a+b\,x^2)^p\,dx\longrightarrow$$

$$\int (f+g\,x)\,(a+b\,x^2)^p\,dx+\int Q_{q-2}[x]\,(a+b\,x^2)^{p+1}\,dx\longrightarrow$$

$$\frac{(a g-b f x)\left(a+b x^2\right)^{p+1}}{2 a b (p+1)}+\frac{1}{2 a (p+1)}\int\left(a+b x^2\right)^{p+1}\left(2 a (p+1) Q_{q-2}[x]+f (2 p+3)\right)dx$$

- **Program code:**

```
Int[Pq_*(a_+b_.*x_^2)^p_,x_Symbol] :=
  With[{Q=PolynomialQuotient[Pq,a+b*x^2,x],
        f=Coeff[PolynomialRemainder[Pq,a+b*x^2,x],x,0],
        g=Coeff[PolynomialRemainder[Pq,a+b*x^2,x],x,1]},
  (a*g-b*f*x)*(a+b*x^2)^(p+1)/(2*a*b*(p+1)) +
  1/(2*a*(p+1))*Int[(a+b*x^2)^(p+1)*ExpandToSum[2*a*(p+1)*Q+f*(2*p+3),x],x]] /;
FreeQ[{a,b},x] && PolyQ[Pq,x] && LtQ[p,-1]
```

5: $\int P_q[x]\left(a+b x^2\right)^p dx$ when $p \nleq -1$

- **Reference: G&R 2.160.3**
- **Derivation: Trinomial recurrence 3a with $A = 0$, $B = 1$ and $m = m - n$**
- **Reference: G&R 2.104**
- **Note: This special case of the Ostrogradskiy-Hermite integration method reduces the degree of the polynomial in the resulting integrand.**
- **Rule 1.1.2.x.5: If $p \nleq -1$, let $e \to P_q[x, q]$, then**

$$\int P_q[x]\left(a+b x^2\right)^p dx \longrightarrow$$

$$\int\left(P_q[x]-e x^q\right)\left(a+b x^2\right)^p dx+e\int x^q\left(a+b x^2\right)^p dx \longrightarrow$$

$$\frac{e x^{q-1}\left(a+b x^2\right)^{p+1}}{b (q+2 p+1)}+\frac{1}{b (q+2 p+1)}\int\left(a+b x^2\right)^p\left(b (q+2 p+1) P_q[x]-a e (q-1) x^{q-2}-b e (q+2 p+1) x^q\right)dx$$

- **Program code:**

```
Int[Pq_*(a_+b_.*x_^2)^p_,x_Symbol] :=
  With[{q=Expon[Pq,x],e=Coeff[Pq,x,Expon[Pq,x]]},
  e*x^(q-1)*(a+b*x^2)^(p+1)/(b*(q+2*p+1)) +
  1/(b*(q+2*p+1))*Int[(a+b*x^2)^p*ExpandToSum[b*(q+2*p+1)*Pq-a*e*(q-1)*x^(q-2)-b*e*(q+2*p+1)*x^q,x],x]] /;
FreeQ[{a,b,p},x] && PolyQ[Pq,x] && Not[LeQ[p,-1]]
```