

Rules for integrands of the form $(a + b \tan[e + f x])^m (c + d \tan[e + f x])^n (A + B \tan[e + f x] + C \tan[e + f x]^2)$

0: $\int (a + b \tan[e + f x])^m (c + d \tan[e + f x])^n (A + B \tan[e + f x] + C \tan[e + f x]^2) dx$ when $bc - ad \neq 0 \wedge Ab^2 - a b B + a^2 C = 0$

Derivation: Algebraic simplification

Basis: If $Ab^2 - a b B + a^2 C = 0$, then $A + B z + C z^2 = \frac{(a+bz)(bB-aC+bCz)}{b^2}$

Rule: If $bc - ad \neq 0 \wedge Ab^2 - a b B + a^2 C = 0$, then

$$\int (a + b \tan[e + f x])^m (c + d \tan[e + f x])^n (A + B \tan[e + f x] + C \tan[e + f x]^2) dx \rightarrow \frac{1}{b^2} \int (a + b \tan[e + f x])^{m+1} (c + d \tan[e + f x])^n (bB - aC + bC \tan[e + f x]) dx$$

Program code:

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Int[(a_.+b_.*tan[e_.+f_.*x_])^m.*(c_.+d_.*tan[e_.+f_.*x_])^n.*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
1/b^2*Int[(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^n*(b*B-a*C+b*C*Tan[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,m,n},x] && NeQ[b*c-a*d,0] && EqQ[A*b^2-a*b*B+a^2*C,0]
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Int[(a_.+b_.*tan[e_.+f_.*x_])^m.*(c_.+d_.*tan[e_.+f_.*x_])^n.*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
-C/b^2*Int[(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^n*(a-b*Tan[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,A,C,m,n},x] && NeQ[b*c-a*d,0] && EqQ[A*b^2+a^2*C,0]
```

1: $\int (a + b \tan[e + f x])^m (c + d \tan[e + f x])^n (A + A \tan[e + f x]^2) dx$

Derivation: Integration by substitution

Basis: $F[\tan[e + f x]] (A + A \tan[e + f x]^2) = \frac{A}{f} \text{Subst}[F[x], x, \tan[e + f x]] \partial_x \tan[e + f x]$

Rule:

$$\int (a + b \tan[e + f x])^m (c + d \tan[e + f x])^n (A + A \tan[e + f x]^2) dx \rightarrow \frac{A}{f} \text{Subst}\left[\int (a + b x)^m (c + d x)^n dx, x, \tan[e + f x]\right]$$

Program code:

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m.*(c_.+d_.*tan[e_.+f_.*x_])^n.*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
A/f*Subst[Int[(a+b*x)^m*(c+d*x)^n,x],x,Tan[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,A,C,m,n},x] && EqQ[A,C]
```

$$2. \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx \text{ when } bc - ad \neq 0 \wedge c^2 + d^2 \neq 0$$

$$1: \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx \text{ when } bc - ad \neq 0 \wedge c^2 + d^2 \neq 0 \wedge n < -1$$

Derivation: Algebraic expansion, nondegenerate tangent recurrence 1c with $c \rightarrow 1$, $d \rightarrow 0$, $A \rightarrow c$, $B \rightarrow d$, $C \rightarrow 0$, $n \rightarrow 0$, $p \rightarrow 0$ and algebraic simplification

$$\text{Basis: } A + Bz + Cz^2 = \frac{c^2 C - Bcd + Ad^2}{d^2} - \frac{(c+dz)(cC - Bd - Cd z)}{d^2}$$

Rule: If $bc - ad \neq 0 \wedge c^2 + d^2 \neq 0 \wedge n < -1$, then

$$\int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx \rightarrow$$

$$\frac{c^2 C - Bcd + Ad^2}{d^2} \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n dx - \frac{1}{d^2} \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^{n+1} (cC - Bd - Cd \tan[ex + f]) dx \rightarrow$$

$$- \frac{(bc - ad) (c^2 C - Bcd + Ad^2) (c + d \tan[ex + f])^{n+1}}{d^2 f (n+1) (c^2 + d^2)} + \frac{1}{d (c^2 + d^2)} \int (c + d \tan[ex + f])^{n+1} .$$

$$(ad (Ac - cC + Bd) + b (c^2 C - Bcd + Ad^2) + d (Abc + aBc - bcC - aAd + bBd + aCd) \tan[ex + f] + bC (c^2 + d^2) \tan[ex + f]^2) dx$$

Program code:

```
Int[(a_.+b_.*tan[e_.+f_.*x_])*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
-(b*c-a*d)*(c^2*C-B*c*d+A*d^2)*(c+d*Tan[e+f*x])^(n+1)/(d^2*f*(n+1)*(c^2+d^2)) +
1/(d*(c^2+d^2))*Int[(c+d*Tan[e+f*x])^(n+1)*
Simp[a*d*(A*c-c*C+B*d)+b*(c^2*C-B*c*d+A*d^2)+d*(A*b*c+a*B*c-b*c*C-a*A*d+b*B*d+a*C*d)*Tan[e+f*x]+b*C*(c^2+d^2)*Tan[e+f*x]^2,x],
FreeQ[{a,b,c,d,e,f,A,B,C},x] && NeQ[b*c-a*d,0] && NeQ[c^2+d^2,0] && LtQ[n,-1]
```

```
Int[(a_.+b_.*tan[e_.+f_.*x_])*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
-(b*c-a*d)*(c^2*C+A*d^2)*(c+d*Tan[e+f*x])^(n+1)/(d^2*f*(n+1)*(c^2+d^2)) +
1/(d*(c^2+d^2))*Int[(c+d*Tan[e+f*x])^(n+1)*
Simp[a*d*(A*c-c*C)+b*(c^2*C+A*d^2)+d*(A*b*c-b*c*C-a*A*d+a*C*d)*Tan[e+f*x]+b*C*(c^2+d^2)*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,C},x] && NeQ[b*c-a*d,0] && NeQ[c^2+d^2,0] && LtQ[n,-1]
```

$$2: \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx \text{ when } bc - ad \neq 0 \wedge c^2 + d^2 \neq 0 \wedge n \neq -1$$

Derivation: Algebraic expansion, nondegenerate tangent recurrence 1b with $c \rightarrow 0$, $d \rightarrow 1$, $A \rightarrow ac$, $B \rightarrow bc + ad$, $C \rightarrow bd$, $m \rightarrow 1 + m$, $n \rightarrow 0$, $p \rightarrow 0$ and algebraic simplification

$$\blacksquare \text{ Basis: } A + Bz + Cz^2 = \frac{c(c+dz)^2}{d^2} - \frac{c^2c - Ad^2 + d(2cC - Bd)z}{d^2}$$

Rule: If $bc - ad \neq 0 \wedge c^2 + d^2 \neq 0 \wedge n \neq -1$, then

$$\int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx \rightarrow$$

$$\frac{C}{d^2} \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^{n+2} dx - \frac{1}{d^2} \int (a + b \tan[ex + f]) (c + d \tan[ex + f])^n (c^2c - Ad^2 + d(2cC - Bd) \tan[ex + f]) dx \rightarrow$$

$$\frac{bC \tan[ex + f] (c + d \tan[ex + f])^{n+1}}{df(n+2)} - \frac{1}{d(n+2)} \int (c + d \tan[ex + f])^n \cdot (bcC - aAd(n+2) - (Ab + aB - bC)d(n+2) \tan[ex + f] - (aCd(n+2) - b(cC - Bd(n+2))) \tan[ex + f]^2) dx$$

Program code:

```
Int[(a_+b_.*tan[e_+f_.*x_])*(c_+d_.*tan[e_+f_.*x_])^n_.*(A_+B_.*tan[e_+f_.*x_]+C_.*tan[e_+f_.*x_]^2),x_Symbol] :=
  b*C*Tan[ex+f]* (c+d*Tan[ex+f])^(n+1)/(d*f*(n+2)) -
  1/(d*(n+2))*Int[(c+d*Tan[ex+f])^n*
  Simp[b*c*C-a*A*d*(n+2)-(A*b+a*B-b*C)*d*(n+2)*Tan[ex+f]-(a*C*d*(n+2)-b*(c*C-B*d*(n+2)))*Tan[ex+f]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,n},x] && NeQ[b*c-a*d,0] && NeQ[c^2+d^2,0] && Not[LtQ[n,-1]]
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Int[(a_+b_.*tan[e_+f_.*x_])*(c_+d_.*tan[e_+f_.*x_])^n_.*(A_+C_.*tan[e_+f_.*x_]^2),x_Symbol] :=
  b*C*Tan[ex+f]* (c+d*Tan[ex+f])^(n+1)/(d*f*(n+2)) -
  1/(d*(n+2))*Int[(c+d*Tan[ex+f])^n*
  Simp[b*c*C-a*A*d*(n+2)-(A*b-b*C)*d*(n+2)*Tan[ex+f]-(a*C*d*(n+2)-b*c*C)*Tan[ex+f]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,n},x] && NeQ[b*c-a*d,0] && NeQ[c^2+d^2,0] && Not[LtQ[n,-1]]
```

3. $\int (a + b \tan[ex + f])^m (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx$ when $bc - ad \neq 0 \wedge a^2 + b^2 = 0$

1: $\int (a + b \tan[ex + f])^m (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx$ when $bc - ad \neq 0 \wedge a^2 + b^2 = 0 \wedge m < 0$

Derivation: Algebraic expansion, singly degenerate tangent recurrence 2b with $A \rightarrow 1, B \rightarrow 0, p \rightarrow 0$ and algebraic simplification

Basis: If $a^2 + b^2 = 0$, then $A + Bz + Cz^2 = \frac{aA+bB-aC}{a} + \frac{(a+bz)(bB-aC+bCz)}{b^2}$

Rule: If $bc - ad \neq 0 \wedge a^2 + b^2 = 0 \wedge m < 0$, then

$$\int (a + b \tan[ex + f])^m (c + d \tan[ex + f])^n (A + B \tan[ex + f] + C \tan[ex + f]^2) dx \rightarrow$$

$$\frac{Ab - aB - bC}{b} \int (a + b \tan[ex + f])^m (c + d \tan[ex + f])^n dx + \frac{1}{b^2} \int (a + b \tan[ex + f])^{m+1} (c + d \tan[ex + f])^n (bB - aC + bC \tan[ex + f]) dx \rightarrow$$

$$\frac{(aA + bB - aC)(a + b \tan[ex + f])^m (c + d \tan[ex + f])^{n+1}}{2fm(bc - ad)} +$$

$$\frac{1}{2am(bc - ad)} \int (a + b \tan[ex + f])^{m+1} (c + d \tan[ex + f])^n \cdot$$

$$(b(c(A+C)m - Bd(n+1)) + a(Bcm + Cd(n+1) - Ad(2m+n+1)) + (bCd(m-n-1) + Abd(m+n+1) + a(2cCm - Bd(m+n+1))) \tan[ex + f] dx$$

Program code:

```
Int[(a+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n.*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(a*A+b*B-a*C)*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(2*f*m*(b*c-a*d)) +
1/(2*a*m*(b*c-a*d))*Int[(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^n*
Simp[b*(c*(A+C)*m-B*d*(n+1))+a*(B*c*m+C*d*(n+1)-A*d*(2*m+n+1))+
(b*C*d*(m-n-1)+A*b*d*(m+n+1)+a*(2*c*C*m-B*d*(m+n+1)))*Tan[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2+b^2,0] && (LtQ[m,0] || EqQ[m+n+1,0])
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Int[(a+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n.*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
a*(A-C)*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(2*f*m*(b*c-a*d)) +
1/(2*a*m*(b*c-a*d))*Int[(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^n*
Simp[b*c*(A+C)*m+a*(C*d*(n+1)-A*d*(2*m+n+1))+(b*C*d*(m-n-1)+A*b*d*(m+n+1)+2*a*c*C*m)*Tan[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2+b^2,0] && (LtQ[m,0] || EqQ[m+n+1,0])
```

$$2. \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge m \neq 0$$

$$1: \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge m \neq 0 \wedge n < -1 \wedge c^2+d^2 \neq 0$$

Derivation: Algebraic expansion and singly degenerate tangent recurrence 1c with $A \rightarrow 1$, $B \rightarrow 0$, $p \rightarrow 0$

$$\text{Basis: } A+Bz+Cz^2 = \frac{c^2C-Bcd+Ad^2}{d^2} - \frac{(c+dz)(cC-Bd-Cdz)}{d^2}$$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge m \neq 0 \wedge n < -1 \wedge c^2+d^2 \neq 0$, then

$$\int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \rightarrow$$

$$\frac{c^2C-Bcd+Ad^2}{d^2} \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n dx - \frac{1}{d^2} \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^{n+1} (cC-Bd-Cd \tan[ex+f]) dx \rightarrow$$

$$\frac{(c^2C-Bcd+Ad^2) (a+b \tan[ex+f])^m (c+d \tan[ex+f])^{n+1}}{df(n+1)(c^2+d^2)} -$$

$$\frac{1}{ad(n+1)(c^2+d^2)} \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^{n+1} \cdot$$

$$(b(c^2C-Bcd+Ad^2)m-ad(n+1)(Ac-cC+Bd)-a(d(Bc-Ad)(m+n+1)-C(c^2m-d^2(n+1))) \tan[ex+f]) dx$$

Program code:

```
Int[(a+b_.*tan[e_.+f_.*x_])^m.*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(c^2*C-B*c*d+A*d^2)*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(n+1)*(c^2+d^2)) -
1/(a*d*(n+1)*(c^2+d^2))*Int[(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)*
Simp[b*(c^2*C-B*c*d+A*d^2)*m-a*d*(n+1)*(A*c-c*C+B*d)-a*(d*(B*c-A*d)*(m+n+1)-C*(c^2*m-d^2*(n+1)))*Tan[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2+b^2,0] && Not[LtQ[m,0]] && LtQ[n,-1] && NeQ[c^2+d^2,0]
```

```
Int[(a+b_.*tan[e_.+f_.*x_])^m.*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(c^2*C+A*d^2)*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(n+1)*(c^2+d^2)) -
1/(a*d*(n+1)*(c^2+d^2))*Int[(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)*
Simp[b*(c^2*C+A*d^2)*m-a*d*(n+1)*(A*c-c*C)-a*(-A*d^2*(m+n+1)-C*(c^2*m-d^2*(n+1)))*Tan[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2+b^2,0] && Not[LtQ[m,0]] && LtQ[n,-1] && NeQ[c^2+d^2,0]
```

2: $\int (a + b \tan[efx])^m (c + d \tan[efx])^n (A + B \tan[efx] + C \tan[efx]^2) dx$ when $bc - ad \neq 0 \wedge a^2 + b^2 = 0 \wedge m \neq 0 \wedge m + n + 1 \neq 0$

Derivation: Algebraic expansion and singly degenerate tangent recurrence 2c with $A \rightarrow c, B \rightarrow d, n \rightarrow n + 1, p \rightarrow 0$

■ **Basis:** $A + Bz + Cz^2 = \frac{c(c+dz)^2}{d^2} + \frac{Ad^2 - c^2C - d(2cC - Bd)z}{d^2}$

— **Rule:** If $bc - ad \neq 0 \wedge a^2 + b^2 = 0 \wedge m \neq 0 \wedge m + n + 1 \neq 0$, then

$$\int (a + b \tan[efx])^m (c + d \tan[efx])^n (A + B \tan[efx] + C \tan[efx]^2) dx \rightarrow$$

$$\frac{c}{d^2} \int (a + b \tan[efx])^m (c + d \tan[efx])^{n+2} dx + \frac{1}{d^2} \int (a + b \tan[efx])^m (c + d \tan[efx])^n (Ad^2 - c^2C - d(2cC - Bd) \tan[efx]) dx \rightarrow$$

$$\frac{C(a + b \tan[efx])^m (c + d \tan[efx])^{n+1}}{df(m+n+1)} +$$

$$\frac{1}{bd(m+n+1)} \int (a + b \tan[efx])^m (c + d \tan[efx])^n (A bd(m+n+1) + C(acm - bd(n+1)) - (Cm(bc - ad) - bBd(m+n+1)) \tan[efx]) dx$$

Program code:

```
Int[(a+b_.*tan[e_.+f_.*x_])^m_.*(c_.+d_.*tan[e_.+f_.*x_])^n_.*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
  C*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(m+n+1)) +
  1/(b*d*(m+n+1))*Int[(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^n*
  Simp[A*b*d*(m+n+1)+C*(a*c*m-b*d*(n+1))-(C*m*(b*c-a*d)-b*B*d*(m+n+1))*Tan[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,m,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2+b^2,0] && Not[LtQ[m,0]] && NeQ[m+n+1,0]
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```
Int[(a+b_.*tan[e_.+f_.*x_])^m_.*(c_.+d_.*tan[e_.+f_.*x_])^n_.*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
  C*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(m+n+1)) +
  1/(b*d*(m+n+1))*Int[(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^n*
  Simp[A*b*d*(m+n+1)+C*(a*c*m-b*d*(n+1))-C*m*(b*c-a*d)*Tan[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,m,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2+b^2,0] && Not[LtQ[m,0]] && NeQ[m+n+1,0]
```

$$4. \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0$$

$$1. \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m > 0$$

$$1: \int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m > 0 \wedge n < -1$$

Derivation: Nondegenerate tangent recurrence 1a with $p \rightarrow 0$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m > 0 \wedge n < -1$, then

$$\int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \rightarrow$$

$$\frac{(A d^2 + c (c C - B d)) (a+b \tan[ex+f])^m (c+d \tan[ex+f])^{n+1}}{d f (n+1) (c^2+d^2)}$$

$$- \frac{1}{d (n+1) (c^2+d^2)} \int (a+b \tan[ex+f])^{m-1} (c+d \tan[ex+f])^{n+1} \cdot$$

$$\left(A d (b d m - a c (n+1)) + (c C - B d) (b c m + a d (n+1)) - \right.$$

$$\left. d (n+1) ((A-C) (b c - a d) + B (a c + b d)) \tan[ex+f] - \right.$$

$$\left. b (d (B c - A d) (m+n+1) - C (c^2 m - d^2 (n+1))) \tan[ex+f]^2 \right) dx$$

Program code:

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(A*d^2+c*(c*C-B*d))*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(n+1)*(c^2+d^2)) -
1/(d*(n+1)*(c^2+d^2))*Int[(a+b*Tan[e+f*x])^(m-1)*(c+d*Tan[e+f*x])^(n+1)*
Simp[A*d*(b*d*m-a*c*(n+1))+c*(c*C-B*d)*(b*c*m+a*d*(n+1)) -
d*(n+1)*((A-C)*(b*c-a*d)+B*(a*c+b*d))*Tan[e+f*x] -
b*(d*(B*c-A*d)*(m+n+1)-C*(c^2*m-d^2*(n+1)))*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && GtQ[m,0] && LtQ[n,-1]
```

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(A*d^2+c^2*C)*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(n+1)*(c^2+d^2)) -
1/(d*(n+1)*(c^2+d^2))*Int[(a+b*Tan[e+f*x])^(m-1)*(c+d*Tan[e+f*x])^(n+1)*
Simp[A*d*(b*d*m-a*c*(n+1))+c*C*(b*c*m+a*d*(n+1)) -
d*(n+1)*((A-C)*(b*c-a*d))*Tan[e+f*x] +
b*(A*d^2*(m+n+1)+C*(c^2*m-d^2*(n+1)))*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,C},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && GtQ[m,0] && LtQ[n,-1]
```

$$2: \int (a+b \tan[efx])^m (c+d \tan[efx])^n (A+B \tan[efx]+C \tan[efx]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m > 0 \wedge n \neq -1$$

Derivation: Nondegenerate tangent recurrence 1b with $p \rightarrow 0$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m > 0 \wedge n \neq -1$, then

$$\int (a+b \tan[efx])^m (c+d \tan[efx])^n (A+B \tan[efx]+C \tan[efx]^2) dx \rightarrow$$

$$\frac{C(a+b \tan[efx])^m (c+d \tan[efx])^{n+1}}{df(m+n+1)} +$$

$$\frac{1}{d(m+n+1)} \int (a+b \tan[efx])^{m-1} (c+d \tan[efx])^n \cdot$$

$$(aAd(m+n+1) - C(bcma+ad(n+1)) + d(Ab+aB-bC)(m+n+1) \tan[efx] - (Cm(bc-ad) - bBd(m+n+1)) \tan[efx]^2) dx$$

Program code:

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m_.*(c_.+d_.*tan[e_.+f_.*x_])^n_*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
C*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(m+n+1)) +
1/(d*(m+n+1))*Int[(a+b*Tan[e+f*x])^(m-1)*(c+d*Tan[e+f*x])^n*
Simp[a*A*d*(m+n+1)-C*(b*c*m+a*d*(n+1))+d*(A*b+a*B-b*C)*(m+n+1)*Tan[e+f*x]-(C*m*(b*c-a*d)-b*B*d*(m+n+1))*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && GtQ[m,0] &&
Not[IGtQ[n,0] && (Not[IntegerQ[m]] || EqQ[c,0] && NeQ[a,0])]
```

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m_.*(c_.+d_.*tan[e_.+f_.*x_])^n_*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
C*(a+b*Tan[e+f*x])^m*(c+d*Tan[e+f*x])^(n+1)/(d*f*(m+n+1)) +
1/(d*(m+n+1))*Int[(a+b*Tan[e+f*x])^(m-1)*(c+d*Tan[e+f*x])^n*
Simp[a*A*d*(m+n+1)-C*(b*c*m+a*d*(n+1))+d*(A*b-b*C)*(m+n+1)*Tan[e+f*x]-C*m*(b*c-a*d)*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && GtQ[m,0] &&
Not[IGtQ[n,0] && (Not[IntegerQ[m]] || EqQ[c,0] && NeQ[a,0])]
```

$$2: \int (a+b \tan[efx])^m (c+d \tan[efx])^n (A+B \tan[efx]+C \tan[efx]^2) dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m < -1$$

Derivation: Nondegenerate tangent recurrence 1c with $p \rightarrow 0$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge m < -1$, then

$$\int (a+b \tan[efx])^m (c+d \tan[efx])^n (A+B \tan[efx]+C \tan[efx]^2) dx \rightarrow$$

$$\frac{(Ab^2 - a(bB - aC)) (a+b \tan[efx])^{m+1} (c+d \tan[efx])^{n+1}}{f(m+1)(bc-ad)(a^2+b^2)} +$$

$$\frac{1}{(m+1)(bc-ad)(a^2+b^2)} \int (a+b \tan[ex+f])^{m+1} (c+d \tan[ex+f])^n \cdot$$

$$\left(A(a(bc-ad)(m+1) - b^2d(m+n+2)) + (bB-aC)(bc(m+1) + ad(n+1)) - \right.$$

$$\left. (m+1)(bc-ad)(Ab-aB-bC) \tan[ex+f] - \right.$$

$$\left. d(Ab^2 - a(bB-aC))(m+n+2) \tan[ex+f]^2 \right) dx$$

Program code:

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(A*b^2-a*(b*B-a*C))*(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^(n+1)/(f*(m+1)*(b*c-a*d)*(a^2+b^2)) +
1/((m+1)*(b*c-a*d)*(a^2+b^2))*Int[(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^n*
Simp[A*(a*(b*c-a*d)*(m+1)-b^2*d*(m+n+2))+(b*B-a*C)*(b*c*(m+1)+a*d*(n+1)) -
(m+1)*(b*c-a*d)*(A*b-a*B-b*C)*Tan[e+f*x] -
d*(A*b^2-a*(b*B-a*C))*(m+n+2)*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && LtQ[m,-1] &&
Not[ILtQ[n,-1] && (Not[IntegerQ[m]] || EqQ[c,0] && NeQ[a,0])]
```

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
(A*b^2+a^2*C)*(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^(n+1)/(f*(m+1)*(b*c-a*d)*(a^2+b^2)) +
1/((m+1)*(b*c-a*d)*(a^2+b^2))*Int[(a+b*Tan[e+f*x])^(m+1)*(c+d*Tan[e+f*x])^n*
Simp[A*(a*(b*c-a*d)*(m+1)-b^2*d*(m+n+2))-a*C*(b*c*(m+1)+a*d*(n+1)) -
(m+1)*(b*c-a*d)*(A*b-b*C)*Tan[e+f*x] -
d*(A*b^2+a^2*C)*(m+n+2)*Tan[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,A,C,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && LtQ[m,-1] &&
Not[ILtQ[n,-1] && (Not[IntegerQ[m]] || EqQ[c,0] && NeQ[a,0])]
```

$$3. \int \frac{(c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2)}{a+b \tan[ex+f]} dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge n \neq 0 \wedge n \neq -1$$

$$1: \int \frac{A+B \tan[ex+f]+C \tan[ex+f]^2}{(a+b \tan[ex+f])(c+d \tan[ex+f])} dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0$$

Derivation: Algebraic expansion

■ **Basis:** $\frac{A+Bz+Cz^2}{(a+bz)(c+dz)} = \frac{a(Ac-cC+Bd)+b(Bc-Ad+Cd)}{(a^2+b^2)(c^2+d^2)} + \frac{(Ab^2-aBb+a^2C)(b-az)}{(bc-ad)(a^2+b^2)(a+bz)} - \frac{(c^2C-Bcd+Ad^2)(d-cz)}{(bc-ad)(c^2+d^2)(c+dz)}$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0$, then

$$\int \frac{A+B \tan[ex+f]+C \tan[ex+f]^2}{(a+b \tan[ex+f])(c+d \tan[ex+f])} dx \rightarrow$$

$$\frac{(a(Ac-cC+Bd)+b(Bc-Ad+Cd))x}{(a^2+b^2)(c^2+d^2)} + \frac{Ab^2-abB+a^2C}{(bc-ad)(a^2+b^2)} \int \frac{b-a \tan[ex+f]}{a+b \tan[ex+f]} dx - \frac{c^2C-Bcd+Ad^2}{(bc-ad)(c^2+d^2)} \int \frac{d-c \tan[ex+f]}{c+d \tan[ex+f]} dx$$

Program code:

```
Int[(A_+B_.*tan[e_+f_.*x_]+C_.*tan[e_+f_.*x_]^2)/((a_+b_.*tan[e_+f_.*x_])*(c_+d_.*tan[e_+f_.*x_])),x_Symbol] :=
(a*(A*c-c*C+B*d)+b*(B*c-A*d+C*d))*x/((a^2+b^2)*(c^2+d^2)) +
(A*b^2-a*b*B+a^2*C)/((b*c-a*d)*(a^2+b^2))*Int[(b-a*Tan[e+f*x])/(a+b*Tan[e+f*x]),x] -
(c^2*C-B*c*d+A*d^2)/((b*c-a*d)*(c^2+d^2))*Int[(d-c*Tan[e+f*x])/(c+d*Tan[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,A,B,C},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0]
```

```
Int[(A_+C_.*tan[e_+f_.*x_]^2)/((a_+b_.*tan[e_+f_.*x_])*(c_+d_.*tan[e_+f_.*x_])),x_Symbol] :=
(a*(A*c-c*C)-b*(A*d-C*d))*x/((a^2+b^2)*(c^2+d^2)) +
(A*b^2+a^2*C)/((b*c-a*d)*(a^2+b^2))*Int[(b-a*Tan[e+f*x])/(a+b*Tan[e+f*x]),x] -
(c^2*C+A*d^2)/((b*c-a*d)*(c^2+d^2))*Int[(d-c*Tan[e+f*x])/(c+d*Tan[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,A,C},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0]
```

$$2: \int \frac{(c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2)}{a+b \tan[ex+f]} dx \text{ when } bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge n \neq 0 \wedge n \neq -1$$

Derivation: Algebraic expansion

$$\blacksquare \text{ Basis: } \frac{A+Bz+Cz^2}{a+bz} = \frac{bB+a(A-C)-(Ab-aB-bC)z}{a^2+b^2} + \frac{(Ab^2-abB+a^2C)(1+z^2)}{(a^2+b^2)(a+bz)}$$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0 \wedge n \neq 0 \wedge n \neq -1$, then

$$\int \frac{(c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2)}{a+b \tan[ex+f]} dx \rightarrow$$

$$\frac{1}{a^2+b^2} \int (c+d \tan[ex+f])^n (bB+a(A-C) + (aB-b(A-C)) \tan[ex+f]) dx +$$

$$\frac{Ab^2-abB+a^2C}{a^2+b^2} \int \frac{(c+d \tan[ex+f])^n (1+\tan[ex+f]^2)}{a+b \tan[ex+f]} dx$$

Program code:

```
Int[(c_+d_.*tan[e_+f_.*x_]^n*(A_+B_.*tan[e_+f_.*x_]+C_.*tan[e_+f_.*x_]^2)/((a_+b_.*tan[e_+f_.*x_])),x_Symbol] :=
1/(a^2+b^2)*Int[(c+d*Tan[e+f*x])^n*Simp[b*B+a*(A-C)+(a*B-b*(A-C))*Tan[e+f*x],x] +
(A*b^2-a*b*B+a^2*C)/(a^2+b^2)*Int[(c+d*Tan[e+f*x])^n*(1+Tan[e+f*x]^2)/(a+b*Tan[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,A,B,C,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && Not[GtQ[n,0]] && Not[LeQ[n,-1]]
```

```
Int[(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+C_.*tan[e_.+f_.*x_]^2)/(a_.+b_.*tan[e_.+f_.*x_],x_Symbol] :=
  1/(a^2+b^2)*Int[(c+d*Tan[e+f*x])^n*Simp[a*(A-C)-(A*b-b*C)*Tan[e+f*x],x],x] +
  (A*b^2+a^2*C)/(a^2+b^2)*Int[(c+d*Tan[e+f*x])^n*(1+Tan[e+f*x]^2)/(a+b*Tan[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,A,C,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0] && Not[GtQ[n,0]] && Not[LeQ[n,-1]]
```

4: $\int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx$ when $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0$

Derivation: Integration by substitution

Basis: $F[\tan[ex+f]] = \frac{1}{f} \text{Subst}\left[\frac{F[x]}{1+x^2}, x, \tan[ex+f]\right] \partial_x \tan[ex+f]$

Rule: If $bc-ad \neq 0 \wedge a^2+b^2 \neq 0 \wedge c^2+d^2 \neq 0$, then

$$\int (a+b \tan[ex+f])^m (c+d \tan[ex+f])^n (A+B \tan[ex+f]+C \tan[ex+f]^2) dx \rightarrow$$

$$\frac{1}{f} \text{Subst}\left[\int \frac{(a+bx)^m (c+dx)^n (A+Bx+Cx^2)}{1+x^2} dx, x, \tan[ex+f]\right]$$

Program code:

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+B_.*tan[e_.+f_.*x_]+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
  ff/f*Subst[Int[(a+b*ff*x)^m*(c+d*ff*x)^n*(A+B*ff*x+C*ff^2*x^2)/(1+ff^2*x^2),x],x,Tan[e+f*x]/ff] /;
FreeQ[{a,b,c,d,e,f,A,B,C,m,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0]
```

```
Int[(a_.+b_.*tan[e_.+f_.*x_])^m*(c_.+d_.*tan[e_.+f_.*x_])^n*(A_.+C_.*tan[e_.+f_.*x_]^2),x_Symbol] :=
  With[{ff=FreeFactors[Tan[e+f*x],x]},
  ff/f*Subst[Int[(a+b*ff*x)^m*(c+d*ff*x)^n*(A+C*ff^2*x^2)/(1+ff^2*x^2),x],x,Tan[e+f*x]/ff] /;
FreeQ[{a,b,c,d,e,f,A,C,m,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2+b^2,0] && NeQ[c^2+d^2,0]
```