

Rules for integrands of the form $(c + d x)^m (a + b \operatorname{Sec}[e + f x])^n$

1. $\int (c + d x)^m (b \operatorname{Sec}[e + f x])^n dx$

1. $\int (c + d x)^m \operatorname{Sec}[e + f x]^n dx$ when $n > 0$

1: $\int (c + d x)^m \operatorname{Sec}[e + f x] dx$ when $m \in \mathbb{Z}^+$

Derivation: Integration by parts

Basis: $\operatorname{Csc}[e + f x] = -\partial_x \frac{2 \operatorname{ArcTanh}[e^{i(e+fx)}]}{f} = \partial_x \frac{\operatorname{Log}[1 - e^{i(e+fx)}]}{f} - \partial_x \frac{\operatorname{Log}[1 + e^{i(e+fx)}]}{f}$

Rule: If $m \in \mathbb{Z}^+$, then

$$\int (c + d x)^m \operatorname{Sec}[e + f x] dx \rightarrow$$

$$-\frac{2 i (c + d x)^m \operatorname{ArcTan}[e^{i(e+fx)}]}{f} - \frac{d m}{f} \int (c + d x)^{m-1} \operatorname{Log}[1 - i e^{i(e+fx)}] dx + \frac{d m}{f} \int (c + d x)^{m-1} \operatorname{Log}[1 + i e^{i(e+fx)}] dx$$

$$\int (c + d x)^m \operatorname{Csc}[e + f x] dx \rightarrow$$

$$-\frac{2 (c + d x)^m \operatorname{ArcTanh}[e^{-i(e+fx)}]}{f} - \frac{d m}{f} \int (c + d x)^{m-1} \operatorname{Log}[1 - e^{-i(e+fx)}] dx + \frac{d m}{f} \int (c + d x)^{m-1} \operatorname{Log}[1 + e^{-i(e+fx)}] dx$$

Program code:

```
Int[(c_.+d_.x_)^m_.csc[e_.+k_.Pi+f_.Complex[0,fz_]x_],x_Symbol] :=
-2*(c+d*x)^m*ArcTanh[E^(-I*k*Pi)*E^(-I*e+f*fz*x)]/(f*fz*I) -
d*m/(f*fz*I)*Int[(c+d*x)^(m-1)*Log[1-E^(-I*k*Pi)*E^(-I*e+f*fz*x)],x] +
d*m/(f*fz*I)*Int[(c+d*x)^(m-1)*Log[1+E^(-I*k*Pi)*E^(-I*e+f*fz*x)],x] /;
FreeQ[{c,d,e,f,fz},x] && IntegerQ[2*k] && IGtQ[m,0]
```

```
Int[(c_.+d_.x_)^m_.csc[e_.+k_.Pi+f_.x_],x_Symbol] :=
-2*(c+d*x)^m*ArcTanh[E^(I*k*Pi)*E^(I*(e+f*x))]/f -
d*m/f*Int[(c+d*x)^(m-1)*Log[1-E^(I*k*Pi)*E^(I*(e+f*x))],x] +
d*m/f*Int[(c+d*x)^(m-1)*Log[1+E^(I*k*Pi)*E^(I*(e+f*x))],x] /;
FreeQ[{c,d,e,f},x] && IntegerQ[2*k] && IGtQ[m,0]
```

```
Int[(c_.+d_.x_)^m_.csc[e_.+f_.Complex[0,fz_]x_],x_Symbol] :=
-2*(c+d*x)^m*ArcTanh[E^(-I*e+f*fz*x)]/(f*fz*I) -
d*m/(f*fz*I)*Int[(c+d*x)^(m-1)*Log[1-E^(-I*e+f*fz*x)],x] +
d*m/(f*fz*I)*Int[(c+d*x)^(m-1)*Log[1+E^(-I*e+f*fz*x)],x] /;
FreeQ[{c,d,e,f,fz},x] && IGtQ[m,0]
```

```
Int[(c_.+d_.*x_)^m_.*csc[e_.+f_.*x_],x_Symbol] :=
-2*(c+d*x)^m*ArcTanh[E^(I*(e+f*x))]/f -
d*m/f*Int[(c+d*x)^(m-1)*Log[1-E^(I*(e+f*x))],x] +
d*m/f*Int[(c+d*x)^(m-1)*Log[1+E^(I*(e+f*x))],x] /;
FreeQ[{c,d,e,f},x] && IGtQ[m,0]
```

$$2. \int (c+dx)^m (b \sec[ex+fx])^n dx \text{ when } n > 1$$

$$1: \int (c+dx)^m \sec[ex+fx]^2 dx \text{ when } m > 0$$

Reference: CRC 430, A&S 4.3.125

Reference: CRC 428, A&S 4.3.121

Basis: $\sec[ex+fx]^2 = \partial_x \frac{\tan[ex+fx]}{f}$

Rule: If $m > 0$, then

$$\int (c+dx)^m \sec[ex+fx]^2 dx \rightarrow \frac{(c+dx)^m \tan[ex+fx]}{f} - \frac{dm}{f} \int (c+dx)^{m-1} \tan[ex+fx] dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*csc[e_.+f_.*x_]^2,x_Symbol] :=
-(c+d*x)^m*Cot[e+f*x]/f +
d*m/f*Int[(c+d*x)^(m-1)*Cot[e+f*x],x] /;
FreeQ[{c,d,e,f},x] && GtQ[m,0]
```

2: $\int (c+dx) (b \sec[ex+fx])^n dx$ when $n > 1 \wedge n \neq 2$

Reference: G&R 2.643.2 with $m \rightarrow 1$, CRC 431, A&S 4.3.126

Reference: G&R 2.643.1 with $m \rightarrow 1$, CRC 429', A&S 4.3.122

Rule: If $n > 1 \wedge n \neq 2$, then

$$\int (c+dx) (b \sec[ex+fx])^n dx \rightarrow \frac{b^2 (c+dx) \tan[ex+fx] (b \sec[ex+fx])^{n-2}}{f(n-1)} - \frac{b^2 d (b \sec[ex+fx])^{n-2}}{f^2 (n-1)(n-2)} + \frac{b^2 (n-2)}{n-1} \int (c+dx) (b \sec[ex+fx])^{n-2} dx$$

Program code:

```
Int[(c_.+d_.*x_)*(b_.*csc[e_.+f_.*x_]^n_,x_Symbol] :=
  -b^2*(c+d*x)*Cot[e+f*x]*(b*Csc[e+f*x])^(n-2)/(f*(n-1)) -
  b^2*d*(b*Csc[e+f*x])^(n-2)/(f^2*(n-1)*(n-2)) +
  b^2*(n-2)/(n-1)*Int[(c+d*x)*(b*Csc[e+f*x])^(n-2),x] /;
FreeQ[{b,c,d,e,f},x] && GtQ[n,1] && NeQ[n,2]
```

$$3: \int (c+dx)^m (b \sec[ex+fx])^n dx \text{ when } n > 1 \wedge n \neq 2 \wedge m > 1$$

Reference: G&R 2.643.2

Reference: G&R 2.643.1

Rule: If $n > 1 \wedge n \neq 2 \wedge m > 1$, then

$$\int (c+dx)^m (b \sec[ex+fx])^n dx \rightarrow \frac{b^2 (c+dx)^m \tan[ex+fx] (b \sec[ex+fx])^{n-2}}{f(n-1)} - \frac{b^2 d m (c+dx)^{m-1} (b \sec[ex+fx])^{n-2}}{f^2 (n-1)(n-2)} + \frac{b^2 (n-2)}{n-1} \int (c+dx)^m (b \sec[ex+fx])^{n-2} dx + \frac{b^2 d^2 m (m-1)}{f^2 (n-1)(n-2)} \int (c+dx)^{m-2} (b \sec[ex+fx])^{n-2} dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_*(b_.*csc[e_.+f_.*x_] )^n_,x_Symbol] :=
-b^2*(c+d*x)^m*Cot[e+f*x]*(b*Csc[e+f*x])^(n-2)/(f*(n-1)) -
b^2*d*m*(c+d*x)^(m-1)*(b*Csc[e+f*x])^(n-2)/(f^2*(n-1)*(n-2)) +
b^2*(n-2)/(n-1)*Int[(c+d*x)^m*(b*Csc[e+f*x])^(n-2),x] +
b^2*d^2*m*(m-1)/(f^2*(n-1)*(n-2))*Int[(c+d*x)^(m-2)*(b*Csc[e+f*x])^(n-2),x] /;
FreeQ[{b,c,d,e,f},x] && GtQ[n,1] && NeQ[n,2] && GtQ[m,1]
```

2. $\int (c+dx)^m (b \sec[ex+fx])^n dx$ when $n < -1$

1: $\int (c+dx) (b \sec[ex+fx])^n dx$ when $n < -1$

Reference: G&R 2.631.3 with $m \rightarrow 1$

Reference: G&R 2.631.2 with $m \rightarrow 1$

Rule: If $n < -1$, then

$$\int (c+dx) (b \sec[ex+fx])^n dx \rightarrow \frac{d (b \sec[ex+fx])^n}{f^2 n^2} - \frac{(c+dx) \sin[ex+fx] (b \sec[ex+fx])^{n+1}}{b f n} + \frac{n+1}{b^2 n} \int (c+dx) (b \sec[ex+fx])^{n+2} dx$$

Program code:

```
Int[(c_.+d_.*x_)*(b_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
d*(b*Csc[e+f*x])^n/(f^2*n^2) +
(c+d*x)*Cos[e+f*x]*(b*Csc[e+f*x])^(n+1)/(b*f*n) +
(n+1)/(b^2*n)*Int[(c+d*x)*(b*Csc[e+f*x])^(n+2),x] /;
FreeQ[{b,c,d,e,f},x] && LtQ[n,-1]
```

$$2: \int (c+dx)^m (b \sec[ex+fx])^n dx \text{ when } n < -1 \wedge m > 1$$

Reference: G&R 2.631.3

Reference: G&R 2.631.2

Rule: If $n < -1 \wedge m > 1$, then

$$\int (c+dx)^m (b \sec[ex+fx])^n dx \rightarrow \frac{d^m (c+dx)^{m-1} (b \sec[ex+fx])^n}{f^2 n^2} - \frac{(c+dx)^m \sin[ex+fx] (b \sec[ex+fx])^{n+1}}{b f n} + \frac{n+1}{b^2 n} \int (c+dx)^m (b \sec[ex+fx])^{n+2} dx - \frac{d^2 m (m-1)}{f^2 n^2} \int (c+dx)^{m-2} (b \sec[ex+fx])^n dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_*(b_.*csc[e_.+f_.**x_]^n_,x_Symbol] :=
d**m*(c+d*x)^(m-1)*(b*Csc[e+f*x])^n/(f^2*n^2) +
(c+d*x)^m*Cos[e+f*x]*(b*Csc[e+f*x])^(n+1)/(b*f*n) +
(n+1)/(b^2*n)*Int[(c+d*x)^m*(b*Csc[e+f*x])^(n+2),x] -
d^2*m*(m-1)/(f^2*n^2)*Int[(c+d*x)^(m-2)*(b*Csc[e+f*x])^n,x] /;
FreeQ[{b,c,d,e,f},x] && LtQ[n,-1] && GtQ[m,1]
```

$$3: \int (c+dx)^m (b \sec[ex+fx])^n dx \text{ when } n \notin \mathbb{Z}$$

Derivation: Piecewise constant extraction

Basis: $\partial_x ((b \cos[ex+fx])^n (b \sec[ex+fx])^n) = 0$

Rule: If $n \notin \mathbb{Z}$, then

$$\int (c+dx)^m (b \sec[ex+fx])^n dx \rightarrow (b \cos[ex+fx])^n (b \sec[ex+fx])^n \int \frac{(c+dx)^m}{(b \cos[ex+fx])^n} dx$$

Program code:

```
Int[(c_.+d_.**x_)^m_*(b_.*csc[e_.+f_.**x_]^n_,x_Symbol] :=
(b*Sin[e+f*x])^n*(b*Csc[e+f*x])^n*Int[(c+d*x)^m/(b*Sin[e+f*x])^n,x] /;
FreeQ[{b,c,d,e,f,m,n},x] && Not[IntegerQ[n]]
```

2: $\int (c+dx)^m (a+b \sec[ex+fx])^n dx$ when $(m|n) \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $(m|n) \in \mathbb{Z}^+$, then

$$\int (c+dx)^m (a+b \sec[ex+fx])^n dx \rightarrow \int (c+dx)^m \text{ExpandIntegrand}[(a+b \sec[ex+fx])^n, x] dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*(a_+b_.*csc[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(c+d*x)^m,(a+b*Csc[e+f*x])^n,x],x] /;
  FreeQ[{a,b,c,d,e,f,m},x] && IGtQ[m,0] && IGtQ[n,0]
```

3: $\int (c+dx)^m (a+b \sec[ex+fx])^n dx$ when $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: If $n \in \mathbb{Z}$, then $(a+b \sec[z])^n = \frac{\cos[z]^{-n}}{(b+a \cos[z])^{-n}}$

Rule: If $n \in \mathbb{Z}^- \wedge m \in \mathbb{Z}^+$, then

$$\int (c+dx)^m (a+b \sec[ex+fx])^n dx \rightarrow \int (c+dx)^m \text{ExpandIntegrand}\left[\frac{\cos[ex+fx]^{-n}}{(b+a \cos[ex+fx])^{-n}}, x\right] dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*(a_+b_.*csc[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandIntegrand[(c+d*x)^m,Sin[e+f*x]^(-n)/(b+a*Sin[e+f*x])^(-n),x],x] /;
  FreeQ[{a,b,c,d,e,f},x] && ILtQ[n,0] && IGtQ[m,0]
```

$$\mathbf{X:} \int (c+dx)^m (a+b \sec[ex+fx])^n dx$$

$$\text{Basis: } \operatorname{Csc}[ex+fx] == \operatorname{Sec}\left[e - \frac{\pi}{2} + fx\right]$$

$$\text{Basis: } \operatorname{Csc}[ex+fx] == i \operatorname{Csch}[ie + ifx]$$

$$\text{Basis: } \operatorname{Csc}[ex+fx] == \operatorname{Sech}\left[i\left(e - \frac{\pi}{2}\right) + ifx\right]$$

Rule:

$$\int (c+dx)^m (a+b \sec[ex+fx])^n dx \rightarrow \int (c+dx)^m (a+b \sec[ex+fx])^n dx$$

Program code:

```
Int[(c_.+d_.*x_)^m_.*csc[e_.+f_.*x_]^n_.,x_Symbol] :=
  If[MatchQ[f,f1_.*Complex[0,j_]],
    If[MatchQ[e,e1_.+Pi/2],
      Unintegrable[(c+d*x)^m*Sech[I*(e-Pi/2)+I*f*x]^n,x],
      (-I)^n*Unintegrable[(c+d*x)^m*Csch[-I*e-I*f*x]^n,x]],
    If[MatchQ[e,e1_.+Pi/2],
      Unintegrable[(c+d*x)^m*Sec[e-Pi/2+f*x]^n,x],
      Unintegrable[(c+d*x)^m*Csc[e+f*x]^n,x]] /;
  FreeQ[{c,d,e,f,m,n},x] && IntegerQ[n]
```

```
Int[(c_.+d_.*x_)^m_.*(a_.+b_.*csc[e_.+f_.*x_] )^n_.,x_Symbol] :=
  Unintegrable[(c+d*x)^m*(a+b*Csc[e+f*x])^n,x] /;
  FreeQ[{a,b,c,d,e,f,m,n},x]
```

$$\mathbf{N:} \int u^m (a+b \sec[v])^n dx \text{ when } u = c+dx \wedge v = ex+fx$$

Derivation: Algebraic normalization

Rule: If $u = c+dx \wedge v = ex+fx$, then

$$\int u^m (a+b \sec[v])^n dx \rightarrow \int (c+dx)^m (a+b \sec[ex+fx])^n dx$$

Program code:

```
Int[u^m_.*(a_.+b_.*Sec[v_])^n_.,x_Symbol] :=
  Int[ExpandToSum[u,x]^m*(a+b*Sec[ExpandToSum[v,x]])^n,x] /;
  FreeQ[{a,b,m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```



```
Int[u_^m_.*(a_.+b_.*Csc[v_])^n_,x_Symbol] :=  
  Int[ExpandToSum[u,x]^m*(a+b*Csc[ExpandToSum[v,x]])^n,x] /;  
FreeQ[{a,b,m,n},x] && LinearQ[{u,v},x] && Not[LinearMatchQ[{u,v},x]]
```