Rules for miscellaneous integrands

Piecewise constant extraction integration rules

x: $\int u\,(c\,x^n)^p\,dx$ when $p\notin\mathbb{Z}$

- **Derivation: Piecewise constant extraction**

- **Basis:** $\partial_x \frac{(c\,x^n)^p}{x^{n\,p}} == 0$

- **Rule: If $p\notin\mathbb{Z}$, then**

$$\int u\,(c\,x^n)^p\,dx \longrightarrow \frac{c^{\text{FracPart}[p]}\,(c\,x^n)^{\text{FracPart}[p]}}{x^{n\,\text{FracPart}[p]}}\int u\,x^{n\,p}\,dx$$

- **Program code:**

```
(* Int[u_.*(c_.*x_^n_)^p_,x_Symbol] :=
  c^FracPart[p]*(c*x^n)^FracPart[p]/x^(n*FracPart[p])*Int[u*x^(n*p),x] /;
FreeQ[{c,n,p},x] && Not[IntegerQ[p]] *)
```

1: $\int u\,(c\,(a+b\,x)^n)^p\,dx$ when $p\notin\mathbb{Z}$

- **Derivation: Piecewise constant extraction**

- **Basis:** $\partial_x \frac{\left(c\,(a+b\,x)^n\right)^p}{(a+b\,x)^{n\,p}} == 0$

- **Basis:** $\frac{\left(c\,(a+b\,x)^n\right)^p}{(a+b\,x)^{n\,p}} == \frac{c^{\text{IntPart}[p]}\left(c\,(a+b\,x)^n\right)^{\text{FracPart}[p]}}{(a+b\,x)^{n\,\text{FracPart}[p]}}$

- **Rule: If $p\notin\mathbb{Z}$, then**

$$\int u\,(c\,(a+b\,x)^n)^p\,dx \longrightarrow \frac{c^{\text{IntPart}[p]}\,(c\,(a+b\,x)^n)^{\text{FracPart}[p]}}{(a+b\,x)^{n\,\text{FracPart}[p]}}\int u\,(a+b\,x)^{n\,p}\,dx$$

- **Program code:**

```
Int[u_*(c_.*(a_.+b_.* x_)^n_)^p_,x_Symbol] :=
  c^IntPart[p]*(c*(a+b*x)^n)^FracPart[p]/(a+b*x)^(n*FracPart[p])*Int[u*(a+b*x)^(n*p),x] /;
FreeQ[{a,b,c,n,p},x] && Not[IntegerQ[p]] && Not[MatchQ[u, x^n1_.*v_. /; EqQ[n,n1+1]]]
```

2: $\int u\,(c\,(d\,(a+b\,x)^n)^p)^q\,dx$ when $p\notin\mathbb{Z}\land q\notin\mathbb{Z}$

Derivation: Piecewise constant extraction

- **Basis:** $\partial_x \frac{(c\,(d\,(a+b\,x)^n)^p)^q}{(a+b\,x)^{n\,p\,q}} == 0$

Note: This should be generalized for arbitrarily deep nesting of powers.

Rule: If $p\notin\mathbb{Z}\land q\notin\mathbb{Z}$**, then**

$$\int u\,(c\,(d\,(a+b\,x)^n)^p)^q\,dx \longrightarrow \frac{(c\,(d\,(a+b\,x)^n)^p)^q}{(a+b\,x)^{n\,p\,q}}\int u\,(a+b\,x)^{n\,p\,q}\,dx$$

Program code:

```
Int[u_.*(c_.*(d_*(a_.+b_.* x_))^p_)^q_,x_Symbol] :=
  (c*(d*(a+b*x))^p)^q/(a+b*x)^(p*q)*Int[u*(a+b*x)^(p*q),x] /;
FreeQ[{a,b,c,d,p,q},x] && Not[IntegerQ[p]] && Not[IntegerQ[q]]
```

```
Int[u_.*(c_.*(d_.*(a_.+b_.* x_)^n_)^p_)^q_,x_Symbol] :=
  (c*(d*(a+b*x)^n)^p)^q/(a+b*x)^(n*p*q)*Int[u*(a+b*x)^(n*p*q),x] /;
FreeQ[{a,b,c,d,n,p,q},x] && Not[IntegerQ[p]] && Not[IntegerQ[q]]
```

Substitution integration rules

1: $\int \frac{\left(a+b\,F\left[c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\right)^n}{A+B\,x+C\,x^2}\,dx$ when $C\,d\,f-A\,e\,g == 0\land B\,e\,g-C\,(e\,f+d\,g) == 0\land n\in\mathbb{Z}^+$

Derivation: Integration by substitution

- **Basis:** $F[x] == 2\,(e\,f-d\,g)\,\text{Subst}\left[\frac{x}{(e-g\,x^2)^2}\,F\left[-\frac{d-f\,x^2}{e-g\,x^2}\right],\,x,\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\partial_x \frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}$

- **Basis: If** $C\,d\,f-A\,e\,g == 0\land B\,e\,g-C\,(e\,f+d\,g) == 0$**, then** $\frac{1}{A+B\,x+C\,x^2} == \frac{2\,e\,g}{C\,(e\,f-d\,g)}\,\text{Subst}\left[\frac{1}{x},\,x,\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\partial_x \frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}$

Rule: If $C\,d\,f-A\,e\,g == 0\land B\,e\,g-C\,(e\,f+d\,g) == 0\land n\in\mathbb{Z}^+$**, then**

$$\int \frac{\left(a+b\,F\left[c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\right)^n}{A+B\,x+C\,x^2}\,dx \longrightarrow \frac{2\,e\,g}{C\,(e\,f-d\,g)}\,\text{Subst}\left[\int \frac{(a+b\,F[c\,x])^n}{x}\,dx,\ x,\ \frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]$$

Program code:

```
Int[(a_.+b_.*F_[c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]])^n_./(A_.+B_.*x_+C_.*x_^2),x_Symbol] :=
  2*e*g/(C*(e*f-d*g))*Subst[Int[(a+b*F[c*x])^n/x,x],x,Sqrt[d+e*x]/Sqrt[f+g*x]] /;
FreeQ[{a,b,c,d,e,f,g,A,B,C,F},x] && EqQ[C*d*f-A*e*g,0] && EqQ[B*e*g-C*(e*f+d*g),0] && IGtQ[n,0]
```

```
Int[(a_.+b_.*F_[c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]])^n_./(A_.+C_.*x_^2),x_Symbol] :=
  2*e*g/(C*(e*f-d*g))*Subst[Int[(a+b*F[c*x])^n/x,x],x,Sqrt[d+e*x]/Sqrt[f+g*x]] /;
FreeQ[{a,b,c,d,e,f,g,A,C,F},x] && EqQ[C*d*f-A*e*g,0] && EqQ[e*f+d*g,0] && IGtQ[n,0]
```

2: $\int \frac{\left(a+b\,F\left[c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\right)^n}{A+B\,x+C\,x^2}\,dx$ when $C\,d\,f-A\,e\,g==0\ \land\ B\,e\,g-C\,(e\,f+d\,g)==0\ \land\ n\notin\mathbb{Z}^+$

Rule: If $C\,d\,f-A\,e\,g==0\ \land\ B\,e\,g-C\,(e\,f+d\,g)==0\ \land\ n\notin\mathbb{Z}^+$**, then**

$$\int \frac{\left(a+b\,F\left[c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\right)^n}{A+B\,x+C\,x^2}\,dx \longrightarrow \int \frac{\left(a+b\,F\left[c\,\frac{\sqrt{d+e\,x}}{\sqrt{f+g\,x}}\right]\right)^n}{A+B\,x+C\,x^2}\,dx$$

Program code:

```
Int[(a_.+b_.*F_[c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]])^n_/(A_.+B_.*x_+C_.*x_^2),x_Symbol] :=
  Unintegrable[(a+b*F[c*Sqrt[d+e*x]/Sqrt[f+g*x]])^n/(A+B*x+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,g,A,B,C,F,n},x] && EqQ[C*d*f-A*e*g,0] && EqQ[B*e*g-C*(e*f+d*g),0] && Not[IGtQ[n,0]]
```

```
Int[(a_.+b_.*F_[c_.*Sqrt[d_.+e_.*x_]/Sqrt[f_.+g_.*x_]])^n_/(A_+C_.*x_^2),x_Symbol] :=
  Unintegrable[(a+b*F[c*Sqrt[d+e*x]/Sqrt[f+g*x]])^n/(A+C*x^2),x] /;
FreeQ[{a,b,c,d,e,f,g,A,C,F,n},x] && EqQ[C*d*f-A*e*g,0] && EqQ[e*f+d*g,0] && Not[IGtQ[n,0]]
```

Derivative divides integration rules

1: $\int \frac{y'[x]}{y[x]} \, dx$

Reference: G&R 2.111.1.2, CRC 27, A&S 3.3.15

Derivation: Integration by substitution and reciprocal rule for integration

Note: Although powerful, this rule is not tried earlier because it is inefficient.

Rule:

$$\int \frac{y'[x]}{y[x]} \, dx \longrightarrow \text{Log}[y[x]]$$

Program code:

```
Int[u_/y_,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
    q*Log[RemoveContent[y,x]] /;
 Not[FalseQ[q]]]
```

```
Int[u_/(y_*w_),x_Symbol] :=
  With[{q=DerivativeDivides[y*w,u,x]},
    q*Log[RemoveContent[y*w,x]] /;
 Not[FalseQ[q]]]
```

2: $\int y'[x]\, y[x]^m \, dx$ when $m \neq -1$

Reference: G&R 2.111.1.1, CRC 23, A&S 3.3.14

Derivation: Integration by substitution and power rule for integration

Note: Although powerful, this rule is not tried earlier because it is inefficient.

Rule: If $m \neq -1$, then

$$\int y'[x]\, y[x]^m \, dx \longrightarrow \frac{y[x]^{m+1}}{m+1}$$

Program code:

```
Int[u_*y_^m_.,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
   q*y^(m+1)/(m+1) /;
 Not[FalseQ[q]]] /;
FreeQ[m,x] && NeQ[m,-1]
```

```
Int[u_*y_^m_.*z_^n_.,x_Symbol] :=
  With[{q=DerivativeDivides[y*z,u*z^(n-m),x]},
   q*y^(m+1)*z^(m+1)/(m+1) /;
 Not[FalseQ[q]]] /;
FreeQ[{m,n},x] && NeQ[m,-1]
```

Algebraic simplification integration rules

1: $\int u \, dx$ when `SimplerIntegrandQ[SimplifyIntegrand[u, x], u, x]`

Derivation: Algebraic simplification

Rule: Let $v =$ `SimplifyIntegrand[u, x]`, **if** `SimplerIntegrandQ[v, u, x]`, **then**

$$\int u \, dx \longrightarrow \int v \, dx$$

Program code:

```
Int[u_,x_Symbol] :=
  With[{v=SimplifyIntegrand[u,x]},
  Int[v,x] /;
 SimplerIntegrandQ[v,u,x]]
```

2. $\int u \left(e \sqrt{a+b x^n} + f \sqrt{c+d x^n}\right)^m dx$ when $m \in \mathbb{Z}^-$

1: $\int u \left(e \sqrt{a+b x^n} + f \sqrt{c+d x^n}\right)^m dx$ when $m \in \mathbb{Z}^- \wedge b e^2 == d f^2$

Derivation: Algebraic simplification

Basis: If $b e^2 == d f^2$, **then** $\frac{1}{e \sqrt{a+b z} + f \sqrt{c+d z}} == \frac{e \sqrt{a+b z} - f \sqrt{c+d z}}{a e^2 - c f^2}$

Rule: If $m \in \mathbb{Z}^- \wedge b e^2 == d f^2$, **then**

$$\int u \left(e \sqrt{a+b x^n} + f \sqrt{c+d x^n}\right)^m dx \longrightarrow \left(a e^2 - c f^2\right)^m \int u \left(e \sqrt{a+b x^n} - f \sqrt{c+d x^n}\right)^{-m} dx$$

Program code:

```
Int[u_.*(e_.*Sqrt[a_.+b_.*x_^n_.]+f_.*Sqrt[c_.+d_.*x_^n_.])^m_,x_Symbol] :=
  (a*e^2-c*f^2)^m*Int[ExpandIntegrand[u*(e*Sqrt[a+b*x^n]-f*Sqrt[c+d*x^n])^(-m),x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && ILtQ[m,0] && EqQ[b*e^2-d*f^2,0]
```

2: $\int u\left(e\sqrt{a+b\,x^n}+f\sqrt{c+d\,x^n}\right)^m dx$ when $m\in\mathbb{Z}^-\wedge a\,e^2==c\,f^2$

Derivation: Algebraic simplification

- **Basis: If** $a\,e^2==c\,f^2$, **then** $\frac{1}{e\sqrt{a+b\,z}+f\sqrt{c+d\,z}}==\frac{e\sqrt{a+b\,z}-f\sqrt{c+d\,z}}{(b\,e^2-d\,f^2)\,z}$

Rule: If $m\in\mathbb{Z}^-\wedge a\,e^2==c\,f^2$, **then**

$$\int u\left(e\sqrt{a+b\,x^n}+f\sqrt{c+d\,x^n}\right)^m dx\longrightarrow\left(b\,e^2-d\,f^2\right)^m\int u\,x^{m\,n}\left(e\sqrt{a+b\,x^n}-f\sqrt{c+d\,x^n}\right)^{-m}dx$$

Program code:

```
Int[u_.*(e_.*Sqrt[a_.+b_.*x_^n_.]+f_.*Sqrt[c_.+d_.*x_^n_.])^m_,x_Symbol] :=
  (b*e^2-d*f^2)^m*Int[ExpandIntegrand[u*x^(m*n)*(e*Sqrt[a+b*x^n]-f*Sqrt[c+d*x^n])^(-m),x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && ILtQ[m,0] && EqQ[a*e^2-c*f^2,0]
```

3: $\int u^m\,(a\,u^n+v)^p\,w\,dx$ when $p\in\mathbb{Z}\wedge n\not>0$

Derivation: Algebraic simplification

Basis: If $p\in\mathbb{Z}$, **then** $(a\,u^n+v)^p==u^{n\,p}\,(a+u^{-n}\,v)^p$

Rule: If $p\in\mathbb{Z}\wedge n\not>0$, **then**

$$\int u^m\,(a\,u^n+v)^p\,w\,dx\longrightarrow\int u^{m+n\,p}\,(a+u^{-n}\,v)^p\,w\,dx$$

Program code:

```
Int[u_^m_.*(a_.*u_^n_+v_)^p_.*w_,x_Symbol] :=
  Int[u^(m+n*p)*(a+u^(-n)*v)^p*w,x] /;
FreeQ[{a,m,n},x] && IntegerQ[p] && Not[GtQ[n,0]] && Not[FreeQ[v,x]]
```

Derivative divides integration rules

1: $\int y'[x]\,(a+b\,y[x])^m\,(c+d\,y[x])^n\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,(a+b\,y[x])^m\,(c+d\,y[x])^n\,dx \longrightarrow \text{Subst}\left[\int (a+b\,x)^m\,(c+d\,x)^n\,dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_*(a_.+b_.*y_)^m_.*(c_.+d_.*v_)^n_.,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
   q*Subst[Int[(a+b*x)^m*(c+d*x)^n,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,c,d,m,n},x] && EqQ[v,y]
```

2: $\int y'[x]\,(a+b\,y[x])^m\,(c+d\,y[x])^n\,(e+f\,y[x])^p\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,(a+b\,y[x])^m\,(c+d\,y[x])^n\,(e+f\,y[x])^p\,dx \longrightarrow \text{Subst}\left[\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_*(a_.+b_.*y_)^m_.*(c_.+d_.*v_)^n_.*(e_.+f_.*w_)^p_.,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
   q*Subst[Int[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && EqQ[v,y] && EqQ[w,y]
```

3: $\int y'[x]\,(a+b\,y[x])^m\,(c+d\,y[x])^n\,(e+f\,y[x])^p\,(g+h\,y[x])^q\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,(a+b\,y[x])^m\,(c+d\,y[x])^n\,(e+f\,y[x])^p\,(g+h\,y[x])^q\,dx \longrightarrow \text{Subst}\left[\int (a+b\,x)^m\,(c+d\,x)^n\,(e+f\,x)^p\,(g+h\,x)^q\,dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_*(a_.+b_.*y_)^m_.*(c_.+d_.*v_)^n_.*(e_.+f_.*w_)^p_.*(g_.+h_.*z_)^q_.,x_Symbol] :=
  With[{r=DerivativeDivides[y,u,x]},
   r*Subst[Int[(a+b*x)^m*(c+d*x)^n*(e+f*x)^p*(g+h*x)^q,x],x,y] /;
 Not[FalseQ[r]]] /;
FreeQ[{a,b,c,d,e,f,g,h,m,n,p,q},x] && EqQ[v,y] && EqQ[w,y] && EqQ[z,y]
```

4: $\int y'[x]\,(a+b\,y[x]^n)^p\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,(a+b\,y[x]^n)^p\,dx \longrightarrow \text{Subst}\left[\int (a+b\,x^n)^p\,dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_.*(a_+b_.*y_^n_),x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
   a*Int[u,x] + b*q*Subst[Int[x^n,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,n},x]
```

```
Int[u_.*(a_.+b_.*y_^n_)^p_,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
   q*Subst[Int[(a+b*x^n)^p,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,n,p},x]
```

5: $\int y'[x]\, y[x]^m\, (a + b\, y[x]^n)^p\, dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\, y[x]^m\, (a + b\, y[x]^n)^p\, dx \longrightarrow \text{Subst}\left[\int x^m\, (a + b\, x^n)^p\, dx,\, x,\, y[x]\right]$$

Program code:

```
Int[u_.*v_^m_.*(a_.+b_.*y_^n_)^p_.,x_Symbol] :=
  Module[{q,r},
    q*r*Subst[Int[x^m*(a+b*x^n)^p,x],x,y] /;
  Not[FalseQ[r=Divides[y^m,v^m,x]]] && Not[FalseQ[q=DerivativeDivides[y,u,x]]]] /;
FreeQ[{a,b,m,n,p},x]
```

6: $\int y'[x]\, \left(a + b\, y[x]^n + c\, y[x]^{2\,n}\right)^p\, dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\, \left(a + b\, y[x]^n + c\, y[x]^{2\,n}\right)^p\, dx \longrightarrow \text{Subst}\left[\int \left(a + b\, x^n + c\, x^{2\,n}\right)^p\, dx,\, x,\, y[x]\right]$$

Program code:

```
Int[u_.*(a_.+b_.*y_^n_+c_.*v_^n2_.)^p_,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
  q*Subst[Int[(a+b*x^n+c*x^(2*n))^p,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && EqQ[v,y]
```

7: $\int y'[x]\,(A+B\,y[x]^n)\,\left(a+b\,y[x]^n+c\,y[x]^{2\,n}\right)^p\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,(A+B\,y[x]^n)\,\left(a+b\,y[x]^n+c\,y[x]^{2\,n}\right)^p\,dx\;\longrightarrow\;\text{Subst}\left[\int (A+B\,x^n)\,\left(a+b\,x^n+c\,x^{2\,n}\right)^p\,dx,\;x,\;y[x]\right]$$

Program code:

```
Int[u_.*(A_+B_.*y_^n_)(a_.+b_.*v_^n_+c_.*w_^n2_.)^p_.,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
  q*Subst[Int[(A+B*x^n)*(a+b*x^n+c*x^(2*n))^p,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,c,A,B,n,p},x] && EqQ[n2,2*n] && EqQ[v,y] && EqQ[w,y]
```

```
Int[u_.*(A_+B_.*y_^n_)(a_.+c_.*w_^n2_.)^p_.,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
  q*Subst[Int[(A+B*x^n)*(a+c*x^(2*n))^p,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,c,A,B,n,p},x] && EqQ[n2,2*n] && EqQ[w,y]
```

8: $\int y'[x]\,y[x]^m\,\left(a+b\,y[x]^n+c\,y[x]^{2\,n}\right)^p\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,y[x]^m\,\left(a+b\,y[x]^n+c\,y[x]^{2\,n}\right)^p\,dx\;\longrightarrow\;\text{Subst}\left[\int x^m\,\left(a+b\,x^n+c\,x^{2\,n}\right)^p\,dx,\;x,\;y[x]\right]$$

Program code:

```
Int[u_.*v_^m_.*(a_.+b_.*y_^n_+c_.*w_^n2_.)^p_.,x_Symbol] :=
  Module[{q,r},
    q*r*Subst[Int[x^m*(a+b*x^n+c*x^(2*n))^p,x],x,y] /;
  Not[FalseQ[r=Divides[y^m,v^m,x]]] && Not[FalseQ[q=DerivativeDivides[y,u,x]]]] /;
FreeQ[{a,b,c,m,n,p},x] && EqQ[n2,2*n] && EqQ[w,y]
```

9: $\int y'[x]\, y[x]^m\, (A + B\, y[x]^n)\, \left(a + b\, y[x]^n + c\, y[x]^{2n}\right)^p\, dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\, y[x]^m\, (A + B\, y[x]^n)\, \left(a + b\, y[x]^n + c\, y[x]^{2n}\right)^p\, dx \longrightarrow \text{Subst}\left[\int x^m\, (A + B\, x^n)\, \left(a + b\, x^n + c\, x^{2n}\right)^p\, dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_.*z_^m_.*(A_+B_.*y_^n_)*(a_.+b_.*v_^n_+c_.*w_^n2_.)^p_.,x_Symbol] :=
  Module[{q,r},
    q*r*Subst[Int[x^m*(A+B*x^n)*(a+b*x^n+c*x^(2*n))^p,x],x,y] /;
  Not[FalseQ[r=Divides[y^m,z^m,x]]] && Not[FalseQ[q=DerivativeDivides[y,u,x]]]] /;
FreeQ[{a,b,c,A,B,m,n,p},x] && EqQ[n2,2*n] && EqQ[v,y] && EqQ[w,y]
```

```
Int[u_.*z_^m_.*(A_+B_.*y_^n_)*(a_.+c_.*w_^n2_.)^p_.,x_Symbol] :=
  Module[{q,r},
    q*r*Subst[Int[x^m*(A+B*x^n)*(a+c*x^(2*n))^p,x],x,y] /;
  Not[FalseQ[r=Divides[y^m,z^m,x]]] && Not[FalseQ[q=DerivativeDivides[y,u,x]]]] /;
FreeQ[{a,c,A,B,m,n,p},x] && EqQ[n2,2*n] && EqQ[w,y]
```

10: $\int y'[x]\, (a + b\, y[x]^n)^m\, (c + d\, y[x]^n)^p\, dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\, (a + b\, y[x]^n)^m\, (c + d\, y[x]^n)^p\, dx \longrightarrow \text{Subst}\left[\int (a + b\, x^n)^m\, (c + d\, x^n)^p\, dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_.*(a_.+b_.*y_^n_)^m_.*(c_.+d_.*v_^n_)^p_.,x_Symbol] :=
  With[{q=DerivativeDivides[y,u,x]},
  q*Subst[Int[(a+b*x^n)^m*(c+d*x^n)^p,x],x,y] /;
 Not[FalseQ[q]]] /;
FreeQ[{a,b,c,d,m,n,p},x] && EqQ[v,y]
```

11: $\int y'[x]\,(a+b\,y[x]^n)^m\,(c+d\,y[x]^n)^p\,(e+f\,y[x]^n)^q\,dx$

Derivation: Integration by substitution

Rule:

$$\int y'[x]\,(a+b\,y[x]^n)^m\,(c+d\,y[x]^n)^p\,(e+f\,y[x]^n)^q\,dx \longrightarrow \text{Subst}\left[\int (a+b\,x^n)^m\,(c+d\,x^n)^p\,(e+f\,x^n)^q\,dx,\ x,\ y[x]\right]$$

Program code:

```
Int[u_.*(a_.+b_.*y_^n_)^m_.*(c_.+d_.*v_^n_)^p_.*(e_.+f_.*w_^n_)^q_.,x_Symbol] :=
  With[{r=DerivativeDivides[y,u,x]},
  r*Subst[Int[(a+b*x^n)^m*(c+d*x^n)^p*(e+f*x^n)^q,x],x,y] /;
 Not[FalseQ[r]]] /;
FreeQ[{a,b,c,d,e,f,m,n,p,q},x] && EqQ[v,y] && EqQ[w,y]
```

12. $\int u\,F^v\,dx$

1: $\int u\,F^v\,dx$ when $\partial_x v == u$

Derivation: Integration by substitution

Rule: If $\partial_x v == u$**, then**

$$\int u\,F^v\,dx \longrightarrow \frac{F^v}{\text{Log}[F]}$$

Program code:

```
Int[u_*F_^v_,x_Symbol] :=
  With[{q=DerivativeDivides[v,u,x]},
   q*F^v/Log[F] /;
 Not[FalseQ[q]]] /;
FreeQ[F,x]
```

2: $\int u\, v^m\, F^v\, dx$ when $\partial_x v == u$

Derivation: Integration by substitution

Rule: If $\partial_x v == u$**, then**

$$\int u\, v^m\, F^v\, dx \longrightarrow \text{Subst}\left[\int x^m\, F^x\, dx,\ x,\ v\right]$$

Program code:

```
Int[u_*w_^m_.*F_^v_,x_Symbol] :=
  With[{q=DerivativeDivides[v,u,x]},
   q*Subst[Int[x^m*F^x,x],x,v] /;
 Not[FalseQ[q]]] /;
FreeQ[{F,m},x] && EqQ[w,v]
```

13. $\int F[f[x]^p,\ g[x]^q]\ f[x]^r\ g[x]^s\ (c\ f'[x]\ g[x] + d\ f[x]\ g'[x])\ dx$

1: $\int u\ (a + b\ v^p\ w^q)^m\ v^r\ w^s\ dx$ when $p\ (s+1) == q\ (r+1)\ \bigwedge\ r \neq -1\ \bigwedge\ \frac{p}{r+1} \in \mathbb{Z}\ \bigwedge\ \text{FreeQ}\left[\frac{u}{p\ w\ \partial_x v + q\ v\ \partial_x w},\ x\right]$

Derivation: Integration by substitution

- **Basis: If** $p\ (s+1) == q\ (r+1)\ \bigwedge\ r \neq -1\ \bigwedge\ \frac{p}{r+1} \in \mathbb{Z}$**, then**
$F[f[x]^p\ g[x]^q]\ f[x]^r\ g[x]^s\ (p\ g[x]\ f'[x] + q\ f[x]\ g'[x]) == \frac{p}{r+1}\ \text{Subst}\left[F\left[x^{\frac{p}{r+1}}\right],\ x,\ f[x]^{r+1}\ g[x]^{s+1}\right]\ \partial_x\left(f[x]^{r+1}\ g[x]^{s+1}\right)$

- **Rule: If** $p\ (s+1) == q\ (r+1)\ \bigwedge\ r \neq -1\ \bigwedge\ \frac{p}{r+1} \in \mathbb{Z}$**, let** $c = \frac{u}{p\ w\ \partial_x v + q\ v\ \partial_x w}$**, if** $\text{FreeQ}[c,\ x]$**, then**

$$\int u\ (a + b\ v^p\ w^q)^m\ v^r\ w^s\ dx \longrightarrow \frac{c\ p}{r+1}\ \text{Subst}\left[\int \left(a + b\ x^{\frac{p}{r+1}}\right)^m dx,\ x,\ v^{r+1}\ w^{s+1}\right]$$

Program code:

```
Int[u_*(a_+b_.*v_^p_.*w_^p_.)^m_.,x_Symbol] :=
  With[{c=Simplify[u/(w*D[v,x]+v*D[w,x])]},
  c*Subst[Int[(a+b*x^p)^m,x],x,v*w] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p},x] && IntegerQ[p]
```

```
Int[u_*(a_+b_.*v_^p_.*w_^q_.)^m_.*v_^r_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]+q*v*D[w,x])]},
  c*p/(r+1)*Subst[Int[(a+b*x^(p/(r+1)))^m,x],x,v^(r+1)*w] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q,r},x] && EqQ[p,q*(r+1)] && NeQ[r,-1] && IntegerQ[p/(r+1)]
```

```
Int[u_*(a_+b_.*v_^p_.*w_^q_.)^m_.*v_^r_.*w_^s_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]+q*v*D[w,x])]},
  c*p/(r+1)*Subst[Int[(a+b*x^(p/(r+1)))^m,x],x,v^(r+1)*w^(s+1)] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q,r,s},x] && EqQ[p*(s+1),q*(r+1)] && NeQ[r,-1] && IntegerQ[p/(r+1)]
```

2: $\int u\,(a\,v^p+b\,w^q)^m\,v^r\,w^s\,dx$ when $p\,(s+1)+q\,(m\,p+r+1)==0\,\bigwedge\,s\neq-1\,\bigwedge\,\frac{q}{s+1}\in\mathbb{Z}\,\bigwedge\,m\in\mathbb{Z}\,\bigwedge\,\text{FreeQ}\left[\frac{u}{p\,w\,\partial_x v-q\,v\,\partial_x w},\,x\right]$

Derivation: Integration by substitution

- **Basis: If** $p\,(s+1)+q\,(m\,p+r+1)==0\,\bigwedge\,s+1\neq0\,\bigwedge\,\frac{q}{s+1}\in\mathbb{Z}\,\bigwedge\,m\in\mathbb{Z}$**, then**

$$(a\,f[x]^p+b\,g[x]^q)^m\,f[x]^r\,g[x]^s\,(p\,g[x]\,f'[x]-q\,f[x]\,g'[x])==$$
$$-\frac{q}{s+1}\,\text{Subst}\left[\left(a+b\,x^{\frac{q}{s+1}}\right)^m,\,x,\,f[x]^{m\,p+r+1}\,g[x]^{s+1}\right]\,\partial_x\left(f[x]^{m\,p+r+1}\,g[x]^{s+1}\right)$$

- **Rule: If** $p\,(s+1)+q\,(m\,p+r+1)==0\,\bigwedge\,s\neq-1\,\bigwedge\,\frac{q}{s+1}\in\mathbb{Z}\,\bigwedge\,m\in\mathbb{Z}$**, let** $c=\frac{u}{p\,w\,\partial_x v-q\,v\,\partial_x w}$**, if** $\text{FreeQ}[c,\,x]$**, then**

$$\int u\,(a\,v^p+b\,w^q)^m\,v^r\,w^s\,dx\,\longrightarrow\,-\frac{c\,q}{s+1}\,\text{Subst}\left[\int\left(a+b\,x^{\frac{q}{s+1}}\right)^m dx,\,x,\,v^{m\,p+r+1}\,w^{s+1}\right]$$

Program code:

```
Int[u_*(a_.*v_^p_.+b_.*w_^q_.)^m_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]-q*v*D[w,x])]},
  c*p*Subst[Int[(b+a*x^p)^m,x],x,v*w^(m*q+1)] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q},x] && EqQ[p+q*(m*p+1),0] && IntegerQ[p] && IntegerQ[m]
```

```
(* Int[u_*(a_.*v_^p_.+b_.*w_^q_.)^m_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]-q*v*D[w,x])]},
  -c*q*Subst[Int[(a+b*x^q)^m,x],x,v^(m*p+1)*w] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q},x] && EqQ[p+q*(m*p+1),0] && IntegerQ[q] && IntegerQ[m] *)
```

```
Int[u_*(a_.*v_^p_.+b_.*w_^q_.)^m_.*v_^r_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]-q*v*D[w,x])]},
  -c*q*Subst[Int[(a+b*x^q)^m,x],x,v^(m*p+r+1)*w] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q,r},x] && EqQ[p+q*(m*p+r+1),0] && IntegerQ[q] && IntegerQ[m]
```

```
Int[u_*(a_.*v_^p_.+b_.*w_^q_.)^m_.*w_^s_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]-q*v*D[w,x])]},
  -c*q/(s+1)*Subst[Int[(a+b*x^(q/(s+1)))^m,x],x,v^(m*p+1)*w^(s+1)] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q,s},x] && EqQ[p*(s+1)+q*(m*p+1),0] && NeQ[s,-1] && IntegerQ[q/(s+1)] && IntegerQ[m]
```

```
Int[u_*(a_.*v_^p_.+b_.*w_^q_.)^m_.*v_^r_.*w_^s_.,x_Symbol] :=
  With[{c=Simplify[u/(p*w*D[v,x]-q*v*D[w,x])]},
  -c*q/(s+1)*Subst[Int[(a+b*x^(q/(s+1)))^m,x],x,v^(m*p+r+1)*w^(s+1)] /;
 FreeQ[c,x]] /;
FreeQ[{a,b,m,p,q,r,s},x] && EqQ[p*(s+1)+q*(m*p+r+1),0] && NeQ[s,-1] && IntegerQ[q/(s+1)] && IntegerQ[m]
```

Substitution integration rules

1: $\int x^m F[x^{m+1}] \, dx$ when $m \neq -1$

Derivation: Integration by substitution

Basis: If $m \neq -1$**, then** $x^m F[x^{m+1}] = \frac{1}{m+1} F[x^{m+1}] \, \partial_x x^{m+1}$

Rule: If $m \neq -1$**, then**

$$\int x^m F[x^{m+1}] \, dx \longrightarrow \frac{1}{m+1} \text{Subst}\left[\int F[x] \, dx, x, x^{m+1}\right]$$

Program code:

```
Int[u_*x_^m_.,x_Symbol] :=
  1/(m+1)*Subst[Int[SubstFor[x^(m+1),u,x],x],x,x^(m+1)] /;
FreeQ[m,x] && NeQ[m,-1] && FunctionOfQ[x^(m+1),u,x]
```

2: $\int F\left[(a+b\,x)^{1/n},\,x\right]\,dx$ when $n\in\mathbb{Z}$

Derivation: Integration by substitution

- **Basis: If** $n\in\mathbb{Z}$, **then** $F\left[(a+b\,x)^{1/n},\,x\right]=\frac{n}{b}\,\text{Subst}\left[x^{n-1}\,F\left[x,\,-\frac{a}{b}+\frac{x^n}{b}\right],\,x,\,(a+b\,x)^{1/n}\right]\,\partial_x\,(a+b\,x)^{1/n}$

Rule: If $n\in\mathbb{Z}$, **then**

$$\int F\left[(a+b\,x)^{1/n},\,x\right]\,dx\;\longrightarrow\;\frac{n}{b}\,\text{Subst}\left[\int x^{n-1}\,F\left[x,\,-\frac{a}{b}+\frac{x^n}{b}\right]\,dx,\,x,\,(a+b\,x)^{1/n}\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=SubstForFractionalPowerOfLinear[u,x]},
  ShowStep["","Int[F[(a+b*x)^(1/n),x],x]",
            "n/b*Subst[Int[x^(n-1)*F[x,-a/b+x^n/b],x],x,(a+b*x)^(1/n)]",Hold[
  lst[[2]]*lst[[4]]*Subst[Int[lst[[1]],x],x,lst[[3]]^(1/lst[[2]])]]] /;
 Not[FalseQ[lst]] && SubstForFractionalPowerQ[u,lst[[3]],x]] /;
SimplifyFlag,

Int[u_,x_Symbol] :=
  With[{lst=SubstForFractionalPowerOfLinear[u,x]},
  lst[[2]]*lst[[4]]*Subst[Int[lst[[1]],x],x,lst[[3]]^(1/lst[[2]])] /;
 Not[FalseQ[lst]] && SubstForFractionalPowerQ[u,lst[[3]],x]]]
```

3: $\int F\left[\left(\frac{a+b x}{c+d x}\right)^{1/n}, x\right] dx$ when $n \in \mathbb{Z}$

Derivation: Integration by substitution

- **Basis: If** $n \in \mathbb{Z}$**, then** $F\left[\left(\frac{a+b x}{c+d x}\right)^{1/n}, x\right] == n\,(b c - a d)\,\text{Subst}\left[\frac{x^{n-1}}{(b-d x^n)^2} F\left[x, \frac{-a+c x^n}{b-d x^n}\right], x, \left(\frac{a+b x}{c+d x}\right)^{1/n}\right] \partial_x \left(\frac{a+b x}{c+d x}\right)^{1/n}$

Rule: If $n \in \mathbb{Z}$**, then**

$$\int F\left[\left(\frac{a+b x}{c+d x}\right)^{1/n}, x\right] dx \longrightarrow n\,(b c - a d)\,\text{Subst}\left[\int \frac{x^{n-1}}{(b-d x^n)^2} F\left[x, \frac{-a+c x^n}{b-d x^n}\right] dx, x, \left(\frac{a+b x}{c+d x}\right)^{1/n}\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=SubstForFractionalPowerOfQuotientOfLinears[u,x]},
  ShowStep["","Int[F[((a+b*x)/(c+d*x))^(1/n),x],x]",
"n*(b*c-a*d)*Subst[Int[x^(n-1)*F[x,(-a+c*x^n)/(b-d*x^n)]/(b-d*x^n)^2,x],x,((a+b*x)/(c+d*x))^(1/n)]",Hold[
  lst[[2]]*lst[[4]]*Subst[Int[lst[[1]],x],x,lst[[3]]^(1/lst[[2]])]]] /;
 Not[FalseQ[lst]]] /;
SimplifyFlag,

Int[u_,x_Symbol] :=
  With[{lst=SubstForFractionalPowerOfQuotientOfLinears[u,x]},
  lst[[2]]*lst[[4]]*Subst[Int[lst[[1]],x],x,lst[[3]]^(1/lst[[2]])] /;
 Not[FalseQ[lst]]]]
```

Piecewise constant extraction integration rules

1: $\int u\,(v^m w^n \cdots)^p\, dx$ when $p \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

- **Basis:** $\partial_x \frac{(a F[x]^m G[x]^n \cdots)^p}{F[x]^{m p} G[x]^{n p} \cdots} == 0$

- **Basis:** $\frac{(a v^m w^n \cdots)^p}{v^{m p} w^{n p} \cdots} == \frac{a^{\text{IntPart}[p]} (a v^m w^n \cdots)^{\text{FracPart}[p]}}{v^{m\,\text{FracPart}[p]} w^{n\,\text{FracPart}[p]} \cdots}$

Rule: If $p \notin \mathbb{Z}$**, then**

$$\int u\,(a\,v^m\,w^n\,\ldots)^p\,dx\;\longrightarrow\;\frac{a^{\text{IntPart}[p]}\,(a\,v^m\,w^n\,\ldots)^{\text{FracPart}[p]}}{v^{m\,\text{FracPart}[p]}\,w^{n\,\text{FracPart}[p]}\,\ldots}\int u\,v^{m\,\text{FracPart}[p]}\,w^{n\,\text{FracPart}[p]}\,\ldots\,dx$$

– **Program code:**

```
Int[u_.*(a_.*v_^m_.*w_^n_.*z_^q_.)^p_,x_Symbol] :=
  a^IntPart[p]*(a*v^m*w^n*z^q)^FracPart[p]/(v^(m*FracPart[p])*w^(n*FracPart[p])*z^(q*FracPart[p]))*Int[u*v^(m*p)*w^(n*p)*z^(p*q),x]
FreeQ[{a,m,n,p,q},x] && Not[IntegerQ[p]] && Not[FreeQ[v,x]] && Not[FreeQ[w,x]] && Not[FreeQ[z,x]]
```

```
Int[u_.*(a_.*v_^m_.*w_^n_.)^p_,x_Symbol] :=
  a^IntPart[p]*(a*v^m*w^n)^FracPart[p]/(v^(m*FracPart[p])*w^(n*FracPart[p]))*Int[u*v^(m*p)*w^(n*p),x] /;
FreeQ[{a,m,n,p},x] && Not[IntegerQ[p]] && Not[FreeQ[v,x]] && Not[FreeQ[w,x]]
```

```
Int[u_.*(a_.*v_^m_.)^p_,x_Symbol] :=
  a^IntPart[p]*(a*v^m)^FracPart[p]/v^(m*FracPart[p])*Int[u*v^(m*p),x] /;
FreeQ[{a,m,p},x] && Not[IntegerQ[p]] && Not[FreeQ[v,x]] && Not[EqQ[a,1] && EqQ[m,1]] && Not[EqQ[v,x] && EqQ[m,1]]
```

2. $\int u\,(a+b\,v^n)^p\,dx$ when $p\notin\mathbb{Z}\,\land\,n\in\mathbb{Z}^-$

1: $\int u\,(a+b\,x^n)^p\,dx$ when $p\notin\mathbb{Z}\,\land\,n\in\mathbb{Z}^-$

Derivation: Piecewise constant extraction

■ **Basis:** $\partial_x\frac{(a+b\,x^n)^p}{x^{n\,p}\left(1+\frac{a\,x^{-n}}{b}\right)^p}=0$

– **Rule: If** $p\notin\mathbb{Z}\,\land\,n\in\mathbb{Z}^-$**, then**

$$\int u\,(a+b\,x^n)^p\,dx\;\longrightarrow\;\frac{b^{\text{IntPart}[p]}\,(a+b\,x^n)^{\text{FracPart}[p]}}{x^{n\,\text{FracPart}[p]}\left(1+\frac{a\,x^{-n}}{b}\right)^{\text{FracPart}[p]}}\int u\,x^{n\,p}\left(1+\frac{a\,x^{-n}}{b}\right)^p dx$$

Program code:

```
Int[u_.*(a_.+b_.*x_^n_)^p_,x_Symbol] :=
  b^IntPart[p]*(a+b*x^n)^FracPart[p]/(x^(n*FracPart[p])*(1+a*x^(-n)/b)^FracPart[p])*Int[u*x^(n*p)*(1+a*x^(-n)/b)^p,x] /;
FreeQ[{a,b,p},x] && Not[IntegerQ[p]] && ILtQ[n,0] && Not[RationalFunctionQ[u,x]] && IntegerQ[p+1/2]
```

2: $\int u\,(a+b\,v^n)^p\,dx$ when $p\notin\mathbb{Z}\,\wedge\,n\in\mathbb{Z}^-$

Derivation: Piecewise constant extraction

- **Basis:** $\partial_x \frac{(a+b\,F[x]^n)^p}{F[x]^{n\,p}\,(b+a\,F[x]^{-n})^p} == 0$
- **Basis:** $\frac{(a+b\,v^n)^p}{v^{n\,p}\,(b+a\,v^{-n})^p} == \frac{(a+b\,v^n)^{\text{FracPart}[p]}}{v^{n\,\text{FracPart}[p]}\,(b+a\,v^{-n})^{\text{FracPart}[p]}}$

Rule: If $p\notin\mathbb{Z}\,\wedge\,n\in\mathbb{Z}^-$**, then**

$$\int u\,(a+b\,v^n)^p\,dx\;\longrightarrow\;\frac{(a+b\,v^n)^{\text{FracPart}[p]}}{v^{n\,\text{FracPart}[p]}\,(b+a\,v^{-n})^{\text{FracPart}[p]}}\int u\,v^{n\,p}\,(b+a\,v^{-n})^p\,dx$$

Program code:

```
Int[u_.*(a_.+b_.*v_^n_)^p_,x_Symbol] :=
  (a+b*v^n)^FracPart[p]/(v^(n*FracPart[p])*(b+a*v^(-n))^FracPart[p])*Int[u*v^(n*p)*(b+a*v^(-n))^p,x] /;
FreeQ[{a,b,p},x] && Not[IntegerQ[p]] && ILtQ[n,0] && BinomialQ[v,x] && Not[LinearQ[v,x]]
```

3: $\int u\,(a+b\,x^m\,v^n)^p\,dx$ when $p\notin\mathbb{Z}\,\wedge\,n\in\mathbb{Z}^-$

Derivation: Piecewise constant extraction

- **Basis:** $\partial_x \frac{(a+b\,x^m\,F[x]^n)^p}{F[x]^{n\,p}\,(b\,x^m+a\,F[x]^{-n})^p} == 0$
- **Basis:** $\frac{(a+b\,x^m\,v^n)^p}{v^{n\,p}\,(b\,x^m+a\,v^{-n})^p} == \frac{(a+b\,x^m\,v^n)^{\text{FracPart}[p]}}{v^{n\,\text{FracPart}[p]}\,(b\,x^m+a\,v^{-n})^{\text{FracPart}[p]}}$

Rule: If $p\notin\mathbb{Z}\,\wedge\,n\in\mathbb{Z}^-$**, then**

$$\int u\,(a+b\,x^m\,v^n)^p\,dx\;\longrightarrow\;\frac{(a+b\,x^m\,v^n)^{\text{FracPart}[p]}}{v^{n\,\text{FracPart}[p]}\,(b\,x^m+a\,v^{-n})^{\text{FracPart}[p]}}\int u\,v^{n\,p}\,(b\,x^m+a\,v^{-n})^p\,dx$$

Program code:

```
Int[u_.*(a_.+b_.*x_^m_.*v_^n_)^p_,x_Symbol] :=
  (a+b*x^m*v^n)^FracPart[p]/(v^(n*FracPart[p])*(b*x^m+a*v^(-n))^FracPart[p])*Int[u*v^(n*p)*(b*x^m+a*v^(-n))^p,x] /;
FreeQ[{a,b,m,p},x] && Not[IntegerQ[p]] && ILtQ[n,0] && BinomialQ[v,x]
```

4: $\int u\,(a\,x^r + b\,x^s)^m\,dx$ when $m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction

- **Basis:** $\partial_x \frac{(a\,x^r+b\,x^s)^m}{x^{m\,r}\,(a+b\,x^{s-r})^m} == 0$
- **Basis:** $\frac{(a\,x^r+b\,x^s)^m}{x^{r\,m}\,(a+b\,x^{s-r})^m} == \frac{(a\,x^r+b\,x^s)^{\text{FracPart}[m]}}{x^{r\,\text{FracPart}[m]}\,(a+b\,x^{s-r})^{\text{FracPart}[m]}}$

Note: This rule should be generalized to handle an arbitrary number of terms.

Rule: If $m \notin \mathbb{Z}$, then

$$\int u\,(a\,x^r + b\,x^s)^m\,dx \longrightarrow \frac{(a\,x^r + b\,x^s)^{\text{FracPart}[m]}}{x^{r\,\text{FracPart}[m]}\,(a + b\,x^{s-r})^{\text{FracPart}[m]}} \int u\,x^{m\,r}\,(a + b\,x^{s-r})^m\,dx$$

Program code:

```
Int[u_.*(a_.*x_^r_.+b_.*x_^s_.)^m_,x_Symbol] :=
  With[{v=(a*x^r+b*x^s)^FracPart[m]/(x^(r*FracPart[m])*(a+b*x^(s-r))^FracPart[m])},
  v*Int[u*x^(m*r)*(a+b*x^(s-r))^m,x] /;
 NeQ[Simplify[v],1]] /;
FreeQ[{a,b,m,r,s},x] && Not[IntegerQ[m]] && PosQ[s-r]
```

Algebraic expansion integration rules

1: $\int \frac{u}{a + b\,x^n}\,dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+$, then

$$\int \frac{u}{a + b\,x^n}\,dx \longrightarrow \int \text{RationalFunctionExpand}\left[\frac{u}{a + b\,x^n},\,x\right] dx$$

Program code:

```
Int[u_/(a_+b_.*x_^n_),x_Symbol] :=
  With[{v=RationalFunctionExpand[u/(a+b*x^n),x]},
  Int[v,x] /;
 SumQ[v]] /;
FreeQ[{a,b},x] && IGtQ[n,0]
```

2. $\int u\,(a+b\,x^n+c\,x^{2\,n})^p\,dx$

1. $\int u\,(a+b\,x^n+c\,x^{2\,n})^p\,dx$ when $b^2-4\,a\,c==0$

1: $\int u\,(a+b\,x^n+c\,x^{2\,n})^p\,dx$ when $b^2-4\,a\,c==0\,\wedge\,p\in\mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b^2-4\,a\,c==0$**, then** $a+b\,z+c\,z^2==\frac{1}{4\,c}\,(b+2\,c\,z)^2$

Rule: If $b^2-4\,a\,c==0\,\wedge\,p\in\mathbb{Z}$**, then**

$$\int u\,(a+b\,x^n+c\,x^{2\,n})^p\,dx\,\longrightarrow\,\frac{1}{4^p\,c^p}\int u\,(b+2\,c\,x^n)^{2\,p}\,dx$$

Program code:

```
Int[u_*(a_.+b_.*x_^n_.+c_.*x_^n2_.)^p_.,x_Symbol] :=
  1/(4^p*c^p)*Int[u*(b+2*c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,n},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && IntegerQ[p] && Not[AlgebraicFunctionQ[u,x]]
```

2: $\int u\,(a+b\,x^n+c\,x^{2\,n})^p\,dx$ when $b^2-4\,a\,c==0\,\wedge\,p\notin\mathbb{Z}$

Derivation: Piecewise constant extraction

Basis: If $b^2-4\,a\,c==0$**, then** $\partial_x\frac{(a+b\,x^n+c\,x^{2\,n})^p}{(b+2\,c\,x^n)^{2\,p}}==0$

Rule: If $b^2-4\,a\,c==0\,\wedge\,p\notin\mathbb{Z}$**, then**

$$\int u\,(a+b\,x^n+c\,x^{2\,n})^p\,dx\,\longrightarrow\,\frac{(a+b\,x^n+c\,x^{2\,n})^p}{(b+2\,c\,x^n)^{2\,p}}\int u\,(b+2\,c\,x^n)^{2\,p}\,dx$$

Program code:

```
Int[u_*(a_.+b_.*x_^n_.+c_.*x_^n2_.)^p_,x_Symbol] :=
  (a+b*x^n+c*x^(2*n))^p/(b+2*c*x^n)^(2*p)*Int[u*(b+2*c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && Not[IntegerQ[p]] && Not[AlgebraicFunctionQ[u,x]]
```

2. $\int u \left(a + b x^n + c x^{2n}\right)^p \, dx$ when $b^2 - 4 a c \neq 0$

1: $\int \frac{u}{a + b x^n + c x^{2n}} \, dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+$, **then**

$$\int \frac{u}{a + b x^n + c x^{2n}} \, dx \longrightarrow \int \text{RationalFunctionExpand}\left[\frac{u}{a + b x^n + c x^{2n}}, x\right] dx$$

Program code:

```
Int[u_/(a_.+b_.*x_^n_.+c_.*x_^n2_.),x_Symbol] :=
  With[{v=RationalFunctionExpand[u/(a+b*x^n+c*x^(2*n)),x]},
  Int[v,x] /;
 SumQ[v]] /;
FreeQ[{a,b,c},x] && EqQ[n2,2*n] && IGtQ[n,0]
```

3: $\int \frac{u}{a x^m + b \sqrt{c x^n}} \, dx$

Derivation: Algebraic simplification

Basis: $\frac{1}{z+w} == \frac{z-w}{z^2-w^2}$

Rule:

$$\int \frac{u}{a x^m + b \sqrt{c x^n}} \, dx \longrightarrow \int \frac{u \left(a x^m - b \sqrt{c x^n}\right)}{a^2 x^{2m} - b^2 c x^n} \, dx$$

Program code:

```
Int[u_./(a_.*x_^m_.+b_.*Sqrt[c_.*x_^n_]),x_Symbol] :=
  Int[u*(a*x^m-b*Sqrt[c*x^n])/(a^2*x^(2*m)-b^2*c*x^n),x] /;
FreeQ[{a,b,c,m,n},x]
```

Substitution integration rules

1: $\int F[a+b\,x]\,dx$

Derivation: Integration by substitution

Basis: $F[a+b\,x] == \frac{1}{b}\,F[a+b\,x]\,\partial_x(a+b\,x)$

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfLinear[u,x]},
  ShowStep["","Int[F[a+b*x],x]","Subst[Int[F[x],x],x,a+b*x]/b",Hold[
  Dist[1/lst[[3]],Subst[Int[lst[[1]],x],x,lst[[2]]+lst[[3]]*x],x]]] /;
 Not[FalseQ[lst]]] /;
SimplifyFlag,

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfLinear[u,x]},
  Dist[1/lst[[3]],Subst[Int[lst[[1]],x],x,lst[[2]]+lst[[3]]*x],x] /;
 Not[FalseQ[lst]]]]
```

2. $\int x^m F[x^n] \, dx$ when $GCD[m+1, n] > 1$

1: $\int \frac{F[(c\,x)^n]}{x} \, dx$

Derivation: Integration by substitution

Basis: $\frac{F[(c\,x)^n]}{x} = \frac{F[(c\,x)^n]}{n\,(c\,x)^n} \partial_x (c\,x)^n$

Rule:

$$\int \frac{F[(c\,x)^n]}{x} \, dx \longrightarrow \frac{1}{n} \text{Subst}\left[\int \frac{F[x]}{x} \, dx, x, (c\,x)^n\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_/x_,x_Symbol] :=
  With[{lst=PowerVariableExpn[u,0,x]},
  ShowStep["","Int[F[(c*x)^n]/x,x]","Subst[Int[F[x]/x,x],x,(c*x)^n]/n",Hold[
  1/lst[[2]]*Subst[Int[NormalizeIntegrand[Simplify[lst[[1]]/x],x],x],x,(lst[[3]]*x)^lst[[2]]]]] /;
 Not[FalseQ[lst]] && NeQ[lst[[2]],0]] /;
SimplifyFlag && NonsumQ[u] && Not[RationalFunctionQ[u,x]],

Int[u_/x_,x_Symbol] :=
  With[{lst=PowerVariableExpn[u,0,x]},
  1/lst[[2]]*Subst[Int[NormalizeIntegrand[Simplify[lst[[1]]/x],x],x],x,(lst[[3]]*x)^lst[[2]]] /;
 Not[FalseQ[lst]] && NeQ[lst[[2]],0]] /;
NonsumQ[u] && Not[RationalFunctionQ[u,x]]]
```

2: $\int x^m F[x^n] \, dx$ when $m \neq -1 \land \mathrm{GCD}[m+1, n] > 1$

Derivation: Integration by substitution

Basis: Let $g = \mathrm{GCD}[m+1, n]$, **then** $x^m F[x^n] = \frac{1}{g} (x^g)^{(m+1)/g-1} F\left[(x^g)^{n/g}\right] \partial_x x^g$

Rule: If $m \neq -1$, **let** $g = \mathrm{GCD}[m+1, n]$, **if** $g > 1$, **then**

$$\int x^m F[x^n] \, dx \longrightarrow \frac{1}{g} \mathrm{Subst}\left[\int x^{(m+1)/g-1} F\left[x^{n/g}\right] dx, x, x^g\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_*x_^m_.,x_Symbol] :=
  With[{lst=PowerVariableExpn[u,m+1,x]},
  ShowStep["If g=GCD[m+1,n]>1,","Int[x^m*F[x^n],x]",
        "1/g*Subst[Int[x^((m+1)/g-1)*F[x^(n/g)],x],x,x^g]",Hold[
  1/lst[[2]]*Subst[Int[NormalizeIntegrand[Simplify[lst[[1]]/x],x],x],x,(lst[[3]]*x)^lst[[2]]]]] /;
 Not[FalseQ[lst]] && NeQ[lst[[2]],m+1]] /;
SimplifyFlag && IntegerQ[m] && NeQ[m,-1] && NonsumQ[u] && (GtQ[m,0] || Not[AlgebraicFunctionQ[u,x]]),

Int[u_*x_^m_.,x_Symbol] :=
  With[{lst=PowerVariableExpn[u,m+1,x]},
  1/lst[[2]]*Subst[Int[NormalizeIntegrand[Simplify[lst[[1]]/x],x],x],x,(lst[[3]]*x)^lst[[2]]] /;
 Not[FalseQ[lst]] && NeQ[lst[[2]],m+1]] /;
IntegerQ[m] && NeQ[m,-1] && NonsumQ[u] && (GtQ[m,0] || Not[AlgebraicFunctionQ[u,x]]))
```

3: $\int x^m F[x]\,dx$ when $m \in \mathbb{F}$

Derivation: Integration by substitution

- **Basis: If** $k \in \mathbb{Z}^+$, **then** $x^m F[x] == k\,(x^{1/k})^{k\,(m+1)-1} F\left[(x^{1/k})^k\right] \partial_x x^{1/k}$

Rule: If $m \in \mathbb{F}$, **let** $k = \text{Denominator}[m]$, **then**

$$\int x^m F[x]\,dx \longrightarrow k\,\text{Subst}\left[\int x^{k\,(m+1)-1} F\left[x^k\right] dx,\ x,\ x^{1/k}\right]$$

Program code:

```
Int[x_^m_*u_,x_Symbol] :=
  With[{k=Denominator[m]},
  k*Subst[Int[x^(k*(m+1)-1)*ReplaceAll[u,x→x^k],x],x,x^(1/k)]] /;
FractionQ[m]
```

4. $\int F\left[\sqrt{a+b\,x+c\,x^2}\,,\,x\right]dx$

1: $\int F\left[\sqrt{a+b\,x+c\,x^2}\,,\,x\right]dx$ when $a>0$

Reference: G&R 2.251.1 (Euler substitution #1)

Derivation: Integration by substitution

Basis: $F\left[\sqrt{a+b\,x+c\,x^2}\,,\,x\right] == 2\,\text{Subst}\left[\frac{c\sqrt{a}-b\,x+\sqrt{a}\,x^2}{(c-x^2)^2}\,F\left[\frac{c\sqrt{a}-b\,x+\sqrt{a}\,x^2}{c-x^2},\,\frac{-b+2\sqrt{a}\,x}{c-x^2}\right],\,x,\,\frac{-\sqrt{a}+\sqrt{a+b\,x+c\,x^2}}{x}\right]\partial_x\frac{-\sqrt{a}+\sqrt{a+b\,x+c\,x^2}}{x}$

Rule: If $a>0$**, then**

$$\int F\left[\sqrt{a+b\,x+c\,x^2}\,,\,x\right]dx \to 2\,\text{Subst}\left[\int\frac{c\sqrt{a}-b\,x+\sqrt{a}\,x^2}{(c-x^2)^2}\,F\left[\frac{c\sqrt{a}-b\,x+\sqrt{a}\,x^2}{c-x^2},\,\frac{-b+2\sqrt{a}\,x}{c-x^2}\right]dx,\,x,\,\frac{-\sqrt{a}+\sqrt{a+b\,x+c\,x^2}}{x}\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfSquareRootOfQuadratic[u,x]},
  ShowStep["","Int[F[Sqrt[a+b*x+c*x^2],x],x]",
            "2*Subst[Int[F[(c*Sqrt[a]-b*x+Sqrt[a]*x^2)/(c-x^2),(-b+2*Sqrt[a]*x)/(c-x^2)]*
                 (c*Sqrt[a]-b*x+Sqrt[a]*x^2)/(c-x^2)^2,x],x,(-Sqrt[a]+Sqrt[a+b*x+c*x^2])/x]",
            Hold[2*Subst[Int[lst[[1]],x],x,lst[[2]]]]] /;
 Not[FalseQ[lst]] && EqQ[lst[[3]],1]] /;
SimplifyFlag && EulerIntegrandQ[u,x],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfSquareRootOfQuadratic[u,x]},
  2*Subst[Int[lst[[1]],x],x,lst[[2]]] /;
 Not[FalseQ[lst]] && EqQ[lst[[3]],1]] /;
EulerIntegrandQ[u,x]]
```

2: $\int F\left[\sqrt{a+b\,x+c\,x^2},\,x\right]dx$ when $a \neq 0 \land c > 0$

Reference: G&R 2.251.2 (Euler substitution #2)

Derivation: Integration by substitution

- **Basis:** $F\left[\sqrt{a+b\,x+c\,x^2},\,x\right] = 2\,\text{Subst}\left[\frac{a\sqrt{c}+b\,x+\sqrt{c}\,x^2}{\left(b+2\sqrt{c}\,x\right)^2}\,F\left[\frac{a\sqrt{c}+b\,x+\sqrt{c}\,x^2}{b+2\sqrt{c}\,x},\,\frac{-a+x^2}{b+2\sqrt{c}\,x}\right],\,x,\,\sqrt{c}\,x+\sqrt{a+b\,x+c\,x^2}\right]\partial_x\left(\sqrt{c}\,x+\sqrt{a+b\,x+c\,x^2}\right)$

Rule: If $a \neq 0 \land c > 0$, then

$$\int F\left[\sqrt{a+b\,x+c\,x^2},\,x\right]dx \to 2\,\text{Subst}\left[\int\frac{a\sqrt{c}+b\,x+\sqrt{c}\,x^2}{\left(b+2\sqrt{c}\,x\right)^2}\,F\left[\frac{a\sqrt{c}+b\,x+\sqrt{c}\,x^2}{b+2\sqrt{c}\,x},\,\frac{-a+x^2}{b+2\sqrt{c}\,x}\right]dx,\,x,\,\sqrt{c}\,x+\sqrt{a+b\,x+c\,x^2}\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfSquareRootOfQuadratic[u,x]},
  ShowStep["","Int[F[Sqrt[a+b*x+c*x^2],x],x]",
            "2*Subst[Int[F[(a*Sqrt[c]+b*x+Sqrt[c]*x^2)/(b+2*Sqrt[c]*x),(-a+x^2)/(b+2*Sqrt[c]*x)]*
                 (a*Sqrt[c]+b*x+Sqrt[c]*x^2)/(b+2*Sqrt[c]*x)^2,x],x,Sqrt[c]*x+Sqrt[a+b*x+c*x^2]]",
            Hold[2*Subst[Int[lst[[1]],x],x,lst[[2]]]]] /;
 Not[FalseQ[lst]] && EqQ[lst[[3]],2]] /;
SimplifyFlag && EulerIntegrandQ[u,x],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfSquareRootOfQuadratic[u,x]},
  2*Subst[Int[lst[[1]],x],x,lst[[2]]] /;
 Not[FalseQ[lst]] && EqQ[lst[[3]],2]] /;
EulerIntegrandQ[u,x]]
```

3: $\int F\left[\sqrt{a+b\,x+c\,x^2},\,x\right]dx$ when $a \not> 0 \land c \not> 0$

Reference: G&R 2.251.3 (Euler substitution #3)

Derivation: Integration by substitution

- **Basis:** $F\left[\sqrt{a+b\,x+c\,x^2},\,x\right] ==$

$$-2\sqrt{b^2-4\,a\,c}\ \mathrm{Subst}\left[\frac{x}{(c-x^2)^2}\,F\left[-\frac{\sqrt{b^2-4\,a\,c}\;x}{c-x^2},\,-\frac{b\,c+c\,\sqrt{b^2-4\,a\,c}+\left(-b+\sqrt{b^2-4\,a\,c}\right)x^2}{2\,c\,(c-x^2)}\right],\,x,\,\frac{2\,c\,\sqrt{a+b\,x+c\,x^2}}{b-\sqrt{b^2-4\,a\,c}+2\,c\,x}\right]\partial_x\frac{2\,c\,\sqrt{a+b\,x+c\,x^2}}{b-\sqrt{b^2-4\,a\,c}+2\,c\,x}$$

Rule: If $a \not> 0 \land c \not> 0$**, then**

$$\int F\left[\sqrt{a+b\,x+c\,x^2},\,x\right]dx \longrightarrow$$

$$-2\sqrt{b^2-4\,a\,c}\ \mathrm{Subst}\left[\int\frac{x}{\left(c-x^2\right)^2}\,F\left[-\frac{\sqrt{b^2-4\,a\,c}\;x}{c-x^2},\,-\frac{b\,c+c\,\sqrt{b^2-4\,a\,c}+\left(-b+\sqrt{b^2-4\,a\,c}\right)x^2}{2\,c\,\left(c-x^2\right)}\right]dx,\,x,\,\frac{2\,c\,\sqrt{a+b\,x+c\,x^2}}{b-\sqrt{b^2-4\,a\,c}+2\,c\,x}\right]$$

Program code:

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfSquareRootOfQuadratic[u,x]},
  ShowStep["","Int[F[Sqrt[a+b*x+c*x^2],x],x]",
           "-2*Sqrt[b^2-4*a*c]*Subst[Int[F[-Sqrt[b^2-4*a*c]*x/(c-x^2),
              (b*c+c*Sqrt[b^2-4*a*c]+(-b+Sqrt[b^2-4*a*c])*x^2)/(-2*c*(c-x^2))]*x/(c-x^2)^2,x],
               x,2*c*Sqrt[a+b*x+c*x^2]/(b-Sqrt[b^2-4*a*c]+2*c*x)]",
           Hold[2*Subst[Int[lst[[1]],x],x,lst[[2]]]]] /;
 Not[FalseQ[lst]] && EqQ[lst[[3]],3]] /;
SimplifyFlag && EulerIntegrandQ[u,x],

Int[u_,x_Symbol] :=
  With[{lst=FunctionOfSquareRootOfQuadratic[u,x]},
  2*Subst[Int[lst[[1]],x],x,lst[[2]]] /;
 Not[FalseQ[lst]] && EqQ[lst[[3]],3]] /;
EulerIntegrandQ[u,x]]
```

Algebraic expansion integration rules

1. $\int \frac{1}{a+b\,v^n}\,dx$ when $n \in \mathbb{Z} \land n > 1$

1. $\int \frac{1}{a+b\,v^n}\,dx$ when $\frac{n}{2} \in \mathbb{Z}^+$

1: $\int \frac{1}{a+b\,v^2}\,dx$

Derivation: Algebraic expansion

Basis: $\frac{1}{a+b\,z^2} == \frac{1}{2\,a\left(1-\frac{z}{\sqrt{-a/b}}\right)} + \frac{1}{2\,a\left(1+\frac{z}{\sqrt{-a/b}}\right)}$

Rule:

$$\int \frac{1}{a+b\,v^2}\,dx \longrightarrow \frac{1}{2\,a}\int \frac{1}{1-\frac{v}{\sqrt{-a/b}}}\,dx + \frac{1}{2\,a}\int \frac{1}{1+\frac{v}{\sqrt{-a/b}}}\,dx$$

Program code:

```
Int[1/(a_+b_.*v_^2),x_Symbol] :=
(*1/(2*a)*Int[Together[1/(1-Rt[-b/a,2]*v)],x] + 1/(2*a)*Int[Together[1/(1+Rt[-b/a,2]*v)],x] /; *)
  1/(2*a)*Int[Together[1/(1-v/Rt[-a/b,2])],x] + 1/(2*a)*Int[Together[1/(1+v/Rt[-a/b,2])],x] /;
FreeQ[{a,b},x]
```

2: $\int \frac{1}{a+b\,v^n}\,dx$ when $\frac{n}{2} \in \mathbb{Z} \bigwedge n > 2$

Derivation: Algebraic expansion

Basis: If $\frac{n}{2} \in \mathbb{Z}^+$**, then** $\frac{1}{a+b\,z^n} == \frac{2}{a\,n}\sum_{k=1}^{n/2} \frac{1}{1-(-1)^{-4\,k/n}\left(-\frac{a}{b}\right)^{-2/n} z^2}$

Rule: If $\frac{n}{2} \in \mathbb{Z} \bigwedge n > 2$**, then**

$$\int \frac{1}{a+b\,v^n}\,dx \longrightarrow \frac{2}{a\,n}\sum_{k=1}^{n/2}\int \frac{1}{1-(-1)^{-4\,k/n}\left(-\frac{a}{b}\right)^{-2/n} v^2}\,dx$$

Program code:

```
Int[1/(a_+b_.*v_^n_),x_Symbol] :=
  Dist[2/(a*n),Sum[Int[Together[1/(1-v^2/((-1)^(4*k/n)*Rt[-a/b,n/2]))],x],{k,1,n/2}],x] /;
FreeQ[{a,b},x] && IGtQ[n/2,1]
```

2: $\int \frac{1}{a + b\, v^n}\, dx$ when $\frac{n-1}{2} \in \mathbb{Z}^+$

Derivation: Algebraic expansion

- **Basis: If** $n \in \mathbb{Z}^+$, **then** $a + b\, z^n == a \prod_{k=1}^{n} \left(1 - (-1)^{-2\,k/n} \left(-\frac{a}{b}\right)^{-1/n} z\right)$
- **Basis: If** $n \in \mathbb{Z}^+$, **then** $\frac{1}{a+b\, z^n} == \frac{1}{a\, n} \sum_{k=1}^{n} \frac{1}{1-(-1)^{-2\,k/n} \left(-\frac{a}{b}\right)^{-1/n} z}$
- **Rule: If** $\frac{n-1}{2} \in \mathbb{Z}^+$, **then**

$$\int \frac{1}{a + b\, v^n}\, dx \longrightarrow \frac{1}{a\, n} \sum_{k=1}^{n} \int \frac{1}{1 - (-1)^{-2\,k/n} \left(-\frac{a}{b}\right)^{-1/n} v}\, dx$$

Program code:

```
Int[1/(a_+b_.*v_^n_),x_Symbol] :=
  Dist[1/(a*n),Sum[Int[Together[1/(1-v/((-1)^(2*k/n)*Rt[-a/b,n]))],x],{k,1,n}],x] /;
FreeQ[{a,b},x] && IGtQ[(n-1)/2,0]
```

2: $\int \frac{P_u}{a + b\, u^n}\, dx$ when $n \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $n \in \mathbb{Z}^+$, **then**

$$\int \frac{P_u}{a + b\, u^n}\, dx \longrightarrow \int \left(\text{ExpandIntegrand}\left[\frac{P_x}{a + b\, x^n},\, x\right] /.\ x \to u\right) dx$$

Program code:

```
Int[v_/(a_+b_.*u_^n_.),x_Symbol] :=
  Int[ReplaceAll[ExpandIntegrand[PolynomialInSubst[v,u,x]/(a+b*x^n),x],x→u],x] /;
FreeQ[{a,b},x] && IGtQ[n,0] && PolynomialInQ[v,u,x]
```

3: $\int u\, dx$ when NormalizeIntegrand[u, x] $\neq$ u

Derivation: Algebraic simplification

Rule: If NormalizeIntegrand[u, x] $\neq$ u**, then**

$$\int u\, dx \longrightarrow \int \text{NormalizeIntegrand}[u, x]\, dx$$

Program code:

```
Int[u_,x_Symbol] :=
  With[{v=NormalizeIntegrand[u,x]},
  Int[v,x] /;
 v=!=u]
```

4: $\int u\, dx$ when ExpandIntegrand[u, x] is a sum

Derivation: Algebraic expansion

Rule: If ExpandIntegrand[u, x] **is a sum, then**

$$\int u\, dx \longrightarrow \int \text{ExpandIntegrand}[u, x]\, dx$$

Program code:

```
Int[u_,x_Symbol] :=
  With[{v=ExpandIntegrand[u,x]},
  Int[v,x] /;
 SumQ[v]]
```

Piecewise constant extraction integration rules

1: $\int u\, (a + b\, x^m)^p\, (c + d\, x^n)^q\, dx$ when $a + d == 0 \land b + c == 0 \land m + n == 0 \land p + q == 0$

Derivation: Piecewise constant extraction

- **Basis:** $\partial_x \frac{(a+b\, x^m)^p}{x^{m\, p} \left(-b-\frac{a}{x^m}\right)^p} == 0$

Rule: If $a + d == 0 \land b + c == 0 \land m + n == 0 \land p + q == 0$

$$\int u\,(a+b\,x^m)^p\,(c+d\,x^n)^q\,dx\;\longrightarrow\;\frac{(a+b\,x^m)^p\,(c+d\,x^n)^q}{x^{m\,p}}\int u\,x^{m\,p}\,dx$$

Program code:

```
Int[u_.*(a_.+b_.*x_^m_.)^p_.*(c_.+d_.*x_^n_.)^q_., x_Symbol] :=
  (a+b*x^m)^p*(c+d*x^n)^q/x^(m*p)*Int[u*x^(m*p),x] /;
FreeQ[{a,b,c,d,m,n,p,q},x] && EqQ[a+d,0] && EqQ[b+c,0] && EqQ[m+n,0] && EqQ[p+q,0]
```

2: $\int u\left(a+b\,x^n+c\,x^{2\,n}\right)^p dx$ when $b^2-4\,a\,c==0\;\bigwedge\;p+\frac{1}{2}\in\mathbb{Z}$

Derivation: Piecewise constant extraction

- **Basis: If** $b^2-4\,a\,c==0\;\bigwedge\;p-\frac{1}{2}\in\mathbb{Z}$**, then** $\left(a+b\,x^n+c\,x^{2\,n}\right)^p==\frac{\sqrt{a+b\,x^n+c\,x^{2\,n}}}{(4\,c)^{p-\frac{1}{2}}\,(b+2\,c\,x^n)}\,(b+2\,c\,x^n)^{2\,p}$
- **Basis: If** $b^2-4\,a\,c==0$**, then** $\partial_x\frac{\sqrt{a+b\,x^n+c\,x^{2\,n}}}{b+2\,c\,x^n}==0$
- **Rule: If** $b^2-4\,a\,c==0\;\bigwedge\;p-\frac{1}{2}\in\mathbb{Z}$**, then**

$$\int u\left(a+b\,x^n+c\,x^{2\,n}\right)^p dx\;\longrightarrow\;\frac{\sqrt{a+b\,x^n+c\,x^{2\,n}}}{(4\,c)^{p-\frac{1}{2}}\,(b+2\,c\,x^n)}\int u\,(b+2\,c\,x^n)^{2\,p}\,dx$$

Program code:

```
Int[u_*(a_+b_.*x_^n_.+c_.*x_^n2_.)^p_, x_Symbol] :=
  Sqrt[a+b*x^n+c*x^(2*n)]/((4*c)^(p-1/2)*(b+2*c*x^n))*Int[u*(b+2*c*x^n)^(2*p),x] /;
FreeQ[{a,b,c,n,p},x] && EqQ[n2,2*n] && EqQ[b^2-4*a*c,0] && IntegerQ[p-1/2]
```

Substitution integration rules

1: $\int F\left[(a+b\,x)^{1/n},\,x\right]\,dx$ when $n\in\mathbb{Z}$

- **Derivation: Integration by substitution**
- **Basis: If** $n\in\mathbb{Z}$**, then** $F\left[(a+b\,x)^{1/n},\,x\right]=\frac{n}{b}\,\text{Subst}\left[x^{n-1}\,F\left[x,\,-\frac{a}{b}+\frac{x^n}{b}\right],\,x,\,(a+b\,x)^{1/n}\right]\,\partial_x\,(a+b\,x)^{1/n}$
- **Rule: If** $n\in\mathbb{Z}$**, then**

$$\int F\left[(a+b\,x)^{1/n},\,x\right]\,dx\;\longrightarrow\;\frac{n}{b}\,\text{Subst}\left[\int x^{n-1}\,F\left[x,\,-\frac{a}{b}+\frac{x^n}{b}\right]\,dx,\,x,\,(a+b\,x)^{1/n}\right]$$

- **Program code:**

```
If[TrueQ[$LoadShowSteps],

Int[u_,x_Symbol] :=
  With[{lst=SubstForFractionalPowerOfLinear[u,x]},
  ShowStep["","Int[F[(a+b*x)^(1/n),x],x]",
            "n/b*Subst[Int[x^(n-1)*F[x,-a/b+x^n/b],x],x,(a+b*x)^(1/n)]",Hold[
  lst[[2]]*lst[[4]]*Subst[Int[lst[[1]],x],x,lst[[3]]^(1/lst[[2]])]]] /;
 Not[FalseQ[lst]]] /;
SimplifyFlag,

Int[u_,x_Symbol] :=
  With[{lst=SubstForFractionalPowerOfLinear[u,x]},
  lst[[2]]*lst[[4]]*Subst[Int[lst[[1]],x],x,lst[[3]]^(1/lst[[2]])] /;
 Not[FalseQ[lst]]]]
```

C: $\int u\,dx$

- **Rule:**

$$\int u\,dx\;\longrightarrow\;\int u\,dx$$

- **Program code:**

```
Int[u_,x_] := CannotIntegrate[u,x]
```